Hopf maps and Wigner's little groups

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Little groups

$$\delta V_\mu = \omega_{\mu\nu} V^\nu, \quad \mu, \nu = 1, \ldots, d$$

$$\delta\lambda = \omega_{\mu\nu} S^{\mu\nu} \lambda, \quad S^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu], \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

The space of non-zero spinors can be represented as a factor space:

$$\{\lambda\} = SO(1, d-1)/\mathcal{G}, \quad g\lambda_0 = 0, \quad g \in \mathcal{G}, \lambda_0 \in \{\lambda\}$$

$$\begin{aligned} \mathcal{G}_{1,3} &= T^2 \\ \mathcal{G}_{1,5} &= SO(3) \ltimes T^4 \\ \mathcal{G}_{1,9} &= SO(7) \ltimes T^8, \end{aligned}$$

Little groups

$$p^\mu = \bar{\lambda}\gamma^\mu\lambda, \quad \bar{\lambda} = \lambda^* A$$

$$\omega_{\mu\nu}p^\nu = 0$$

$$\begin{cases} \omega_{0i}p^i = 0 \\ \omega_{i0}p^0 + \omega_{ij}p^j = 0 \end{cases} \Rightarrow \omega_{0i} = -\omega_{i0} = \frac{\omega_{ij}p^j}{p_0}$$

$$\begin{aligned} d = 3+1: &\quad SO(2) \ltimes T^2 \\ d = 5+1 &\quad SO(4) \ltimes T^4 \\ d = 9+1 &\quad SO(8) \ltimes T^8 \end{aligned}$$

Little groups and Hopf maps

Transformations which change λ but do not change p

$$SO(2) \otimes T^2/T^2 = SO(2) \equiv S^1$$
$$SO(4) \otimes T^4/SO(3) \otimes T^4 = SO(3) \equiv S^3$$
$$SO(8) \otimes T^8/SO(7) \otimes T^8 = S^7$$

$$(\lambda^*)^T \lambda = p_i p_i = 1, \quad i = 1, \ldots, d-1$$

$$S^3/S^1 = S^2, \qquad S^7/S^3 = S^4 \qquad S^{15}/S^7 = S^8,$$

Hopf maps and normed division algebras

$$\mathbf{p} = 2\bar{\mathbf{u}}_1\mathbf{u}_2, \quad p_{n+1} = \bar{\mathbf{u}}_1\mathbf{u}_1 - \bar{\mathbf{u}}_2\mathbf{u}_2, \quad n = 1, 2, 4, 8$$

$\mathbf{u}_1, \mathbf{u}_2$ - real, complex, quaternionic or octonionic numbers.

$$p_0^2 \equiv \bar{\mathbf{p}}\mathbf{p} + p_{n+1}^2 = (\bar{\mathbf{u}}_1\mathbf{u}_1 + \bar{\mathbf{u}}_2\mathbf{u}_2)^2$$

Inverse formulae:

$$\mathbf{u}_\alpha = \mathbf{g} r_\alpha,$$

$$r_1 = \sqrt{\frac{p_0 + p_{p+1}}{2}}, \quad \mathbf{r}_2 \equiv \mathbf{r}_+ = \frac{\mathbf{p}}{\sqrt{2(p_0 + p_{p+1})}}, \quad , \quad \bar{\mathbf{g}}\mathbf{g} = 1$$

Hopf maps and normed division algebras

The transformation $\mathbf{g} \mapsto \tau\mathbf{g}$ does not change the quantities p. $\tau\bar{\tau} = 1$

For real, complex and quaternionic numbers:

$$\mathbf{u}_\alpha \to \tau\mathbf{u}_\alpha$$

For octonions:

$$\mathbf{u}_\alpha = \frac{(\tau\mathbf{u}_1)(\bar{\mathbf{u}}_1\mathbf{u}_\alpha)}{\bar{\mathbf{u}}_1\mathbf{u}_1}$$

Hopf maps:

$$S^{2n-1}/S^{n-1} = S^n$$

d=3+1

$$p^\mu = \lambda^* \gamma^0 \gamma^\mu \lambda$$

$$p^0 = Z\bar{Z}, \quad p^i = Z\sigma^i \bar{Z}, \quad \lambda = (Z, \sigma^1 \bar{Z}) \quad Z = (z_1, z_2), \quad \bar{Z} = Z^*$$

Solution for δp=0:

$$\omega_{0i} = \epsilon_{ijk} a^j p^k, \quad \omega_{ij} = \epsilon_{ijk} a^k p^0,$$

$$\delta Z = 2\omega_{0i}\sigma^i Z + \omega_{ij}\epsilon^{ijk}\sigma_k Z$$

After applying Fierz reordering formula the variation of Z looks:

$$\delta Z = \imath a^i \, (Z\sigma_i \bar{Z}) Z$$

d=5+1

$$p^\mu = \bar{\lambda}\Gamma^\mu\lambda \qquad\qquad p^0 = Z\bar{Z}, \quad p^i = -\bar{Z}\gamma^i Z$$

Solution for δp=0:

$$\omega_{0i} = -\omega_{i0} = \frac{\omega_{ij}p^j}{p_0}$$

$$\delta Z = -2\frac{\omega_{ij}(\bar{Z}\gamma^j Z)}{p_0}\gamma^i Z + \omega_{ij}\gamma^{ij} Z$$

After applying Fierz reordering formula the variation of Z looks:

$$\delta Z = -(\bar{Z}\gamma^{ij}Z)\omega_{ij}Z + (ZC\gamma^{ij}Z)\omega_{ij}C\bar{Z}$$

d=5+1. Connection with quaternions

Let us define:

$$\mathbf{u}_1 = z_1 + z_2\mathbf{j}, \quad \mathbf{u}_2 = z_3 + z_4\mathbf{j},$$

$$\tau = (\bar{Z}\gamma^{ij}Z)\omega_{ij} + (ZC\gamma^{ij}Z)\omega_{ij}\mathbf{j}$$

The transformation of spinor elements can be given:

$$\delta\mathbf{u}_\alpha = \tau\mathbf{u}_\alpha$$

τ is truly imaginary quaternion. So the transformation of $\boldsymbol{u}_{1},$ $\boldsymbol{u}_2$ is an infinitesimal from the

$$\delta\mathbf{u}_\alpha = \mathbf{G}\mathbf{u}_\alpha, \quad \mathbf{G}\bar{\mathbf{G}} = 1$$

d=9+1

$$p^\mu = \bar{\lambda}\Gamma^\mu\lambda \qquad\qquad p^0 = Z\bar{Z}, \quad p^i = -\bar{Z}\gamma^i Z$$

Solution for δp=0:

$$\omega_{0i} = -\omega_{i0} = \frac{\omega_{ij}p^j}{p_0}$$

$$\delta Z = -2\frac{\omega_{ij}(\bar{Z}\gamma^j Z)}{p_0}\gamma^i Z + \omega_{ij}\gamma^{ij} Z$$

After applying Fierz reordering formula the variation of Z looks:

$$\delta Z = -\frac{1}{6}\omega_{ij}\left(ZC\gamma^{ijlm}Z\right)\gamma^{lm}Z$$