Deformed Calogero-Moser systems and Lie superalgebras

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Plan

- **Calogero-Moser problem and its generalisations**
- **Lie superalgebras and generalised root systems**
- **Deformed CM operators**
- **View from infinity**
- **Back to Lie superalgebras**

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References

A.N. Sergeev, A.P. Veselov Comm. Math. Phys. 245 (2004), 249-278
A.N. Sergeev, A.P. Veselov Adv. Math. 222 (2009),1687-1726
A.N. Sergeev, A.P. Veselov arXiv:0905.2603

Calogero-Moser problem and its generalisations

Calogero (1971), Sutherland (1971), Moser (1975): Interacting particles on the line with the potential

$$U(x_1, \dots, x_n) = \sum_{1 \le i < j \le n} \frac{g^2 \omega^2}{\sinh^2 \omega(x_i - x_j)}$$

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Olshanetsky and Perelomov (1976) : Generalisations related to root systems

$$L = -\Delta + \sum_{\alpha \in R_+} \frac{m_\alpha (m_\alpha + 2m_{2\alpha} + 1)(\alpha, \alpha)}{\sinh^2(\alpha, x)}$$

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$$L=-\Delta+\sum_{\alpha\in R_+}\frac{m_\alpha(m_\alpha+2m_{2\alpha}+1)(\alpha,\alpha)}{\sinh^2(\alpha,x)}$$

Remark. If R is a root system of a compact symmetric space X then L is conjugated to the radial part of the Laplace-Beltrami operator on X

$$\mathcal{L}=-\Delta+2\sum_{\alpha\in R_+}m_\alpha\cot(\alpha,x)\partial_\alpha :$$

$$\mathcal{L}=\hat{\psi}_0^{-1}\circ L\circ\hat{\psi}_0+const,\quad \psi_0=\prod_{\alpha\in R_+}\sin^{-m_\alpha}(\alpha,x).$$

O. Chalykh, M. Feigin, A.V. (1996), A.N. Sergeev (2000)

$$L^A_{m,n}(k) = -\Delta_x - k\Delta_y + \sum_{i<j}^{m} \frac{2k(k+1)}{\sinh^2(x_i - x_j)} + \sum_{i<j}^{n} \frac{2(k^{-1}+1)}{\sinh^2(y_i - y_j)} + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{2(k+1)}{\sinh^2(x_i - y_j)},$$

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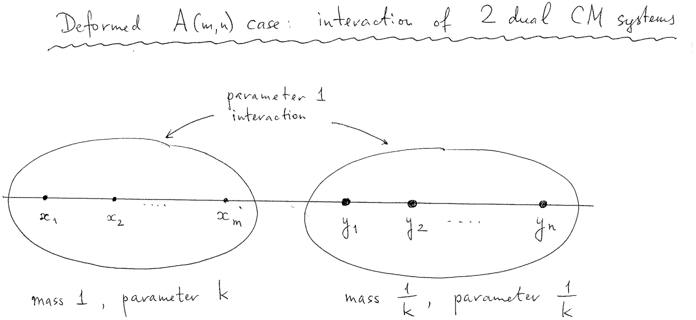

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1) R spans V and $R = -R$;

2) if $\alpha, \beta \in R$ and $< \alpha, \alpha > \neq 0$ then $\frac{2<\alpha,\beta>}{<\alpha,\alpha>} \in \mathbf{Z}$ and
$s_\alpha(\beta) = \beta - \frac{2<\alpha,\beta>}{<\alpha,\alpha>}\alpha \in R$;

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3) if $\alpha \in R$ and $< \alpha, \alpha >= 0$ then for any $\beta \in R$ such that $< \alpha, \beta > \neq 0$ at least one of the vectors $\beta + \alpha$ or $\beta - \alpha$ belongs to R.

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Remark. There is no Weyl group related to GRS because one can not define a reflection wrt isotropic root. There exists only a partial symmetry group W_0 generated by reflections wrt the non-isotropic roots.

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classical series $A(n, m)$ and $BC(n, m)$ and three exceptional cases $G(1, 2)$, $AB(1, 3)$, $D(2, 1, \lambda)$

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Classical series:

$A(n-1, m-1)$ (corresponding to Lie superalgebra $sl(n,m)$):

$$R = \{e_i - e_j, i \neq j, 1 \leq i, j \leq n+m\}$$

$$B(u,v) = \sum_{i=1}^{n} u^i v^i - \sum_{j=n+1}^{n+m} u^j v^j.$$

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In $BC(n,m)$ case (including orthosymplectic Lie superalgebras $\mathfrak{osp}(2m+1,2n)$ and $\mathfrak{osp}(2m,2n)$) the form B is the same and

$$R = \{\pm e_i, : \ \pm 2e_i, : \ \pm e_i \pm e_j, : 1 \leq i < j \leq n+m\}.$$

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where the bilinear form and multiplicities are deformed in such a way that

1) new form B and multiplicities are W_0-invariant;

2) all isotropic roots have multiplicity 1;

3) existence of "radial gauge":

$$\psi_0 = \prod_{\alpha \in R_+} \sin^{-m_\alpha}(\alpha, x)$$

is a (pseudo)ground state of L: $L\psi_0 = \kappa\psi_0$.

Deformed $A(m,n)$ root system

In $A(n-1,m-1)$ case the multiplicities are k for $A(n-1)$, k^{-1} for $A(m-1)$,

$$B(u,v) = \sum_{i=1}^{n} u^i v^i + k \sum_{j=n+1}^{n+m} u^j v^j$$

and the corresponding CM operator was given above.

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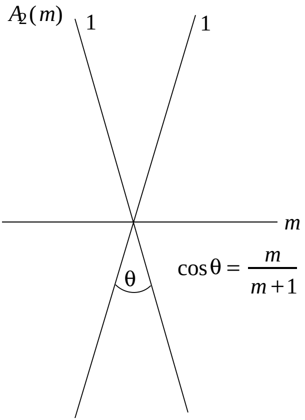

Deformed $BC(m, n)$ case

In $BC_{n,m}$ case

$$R = \{\pm e_i, : \ \pm 2e_i, : \ \pm e_i \pm e_j, : 1 \leq i < j \leq n + m\}$$

B is the same as above, multiplicities are

$$m(e_i \pm e_j) = k, \quad m(e_i) = p, \quad m(2e_i) = q, i, j = 1, \dots, n,$$
$$m(e_i \pm e_j) = k^{-1}, \quad m(e_j) = r, \quad m(2e_j) = s, i, j = n + 1, \dots, n + m,$$

where p, q, r, s are satisfying the relations

$$p = kr, \quad 2q + 1 = k(2s + 1).$$

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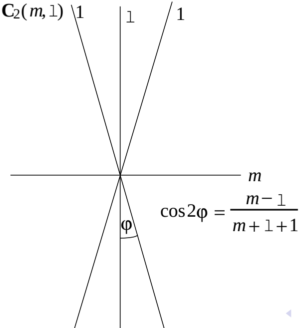

View from infinity: symmetric functions

Let $P_N = \mathbb{C}[x_1, \ldots, x_N]$ be the polynomial algebra in N independent variables and

$$\Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N} \subset P_N$$

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Power sums

$$p_k = x_1^k + x_2^k + \ldots, \quad k = 1, 2, \ldots$$

are **free** algebraic generators of Λ with degrees $deg\ p_k = k$:

Any symmetric function $f \in \Lambda^r$ is a polynomial of $p_1, \ldots, p_r$.

is defined as a unique differential operator $\mathcal{L}^{(\infty)}_{k,p_0} : \Lambda \to \Lambda$, such that for all $N = 1, 2, \ldots$ and $p_0 = N$ the following diagram is commutative

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathcal{L}^{(\infty)}_{k,p_0}} & \Lambda \\ \downarrow \varphi_N & & \downarrow \varphi_N \\ \Lambda_N & \xrightarrow{\mathcal{L}^{(N)}_{k}} & \Lambda_N \end{array}$$

where

$$\mathcal{L}^{(N)}_k = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} \right)^2 - k \sum_{i<j}^{N} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right).$$

is the usual CM operator in exponential coordinates and radial gauge.

CM operator in infinite dimension

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where

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Explicitly:

$$\mathcal{L}^{(\infty)}_{k,p_0} = \sum_{a,b>0} p_{a+b} \partial_a \partial_b - k \sum_{a,b>0} p_a p_b \partial_{a+b} - k p_0 \sum_{a>0} p_a \partial_a + (1+k) \sum_{a>0} a p_a \partial_a,$$

where $\partial_a = a \frac{\partial}{\partial p_a}$.

Duality

Stanley, 1989: $k \longrightarrow k^{-1}$

$$\theta^{-1} \circ \mathcal{L}^{(\infty)}_{k,p_0} \circ \theta = k\mathcal{L}^{(\infty)}_{k^{-1},k^{-1}p_0}, \quad \theta : p_a \longrightarrow kp_a.$$

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For corresponding eigenfunctions (**Jack symmetric functions**)

$$\theta(P_\lambda(z,k)) = c(\lambda,k)P_{\lambda'}(z,1/k),$$

where λ' is the transposed Young diagram.

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Remark. This duality **can not be seen at finite-dimensional level** !

Deformed CM operator as restriction

Let $\Lambda_{m,n,k} \subset \mathbb{C}[u_1, \ldots, u_m, v_1, \ldots, v_n]^{S_m \times S_n}$ consists of polynomials, satisfying

$$\left(u_i \frac{\partial}{\partial u_i} - k v_j \frac{\partial}{\partial v_j} \right) f = 0$$

on the hyperplane $u_i = v_j$. Consider the homomorphism $\varphi_{m,n} : \Lambda \to \Lambda_{m,n,k}$

$$\varphi_{m,n}(p_a) = p_a(u, v, k) = \sum_{i=1}^{m} u_i^a + k^{-1} \sum_{\alpha=1}^{n} v_\alpha^a.$$

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Theorem 1. The following diagram with $p_0 = m + kn$ is commutative

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathcal{L}^{(\infty)}_{k,p_0}} & \Lambda \\ \downarrow \varphi_{m,n} & & \downarrow \varphi_{m,n} \\ \Lambda_{m,n,k} & \xrightarrow{L^A_{m,n}(k)} & \Lambda_{m,n,k} \end{array}$$

where $L^A_{m,n}(k)$ is the (gauged) deformed CM operator.

BC_N operator $L^N_{k,p,q} =$

$$\Delta_N - \sum_{i<j}^{N} \left(\frac{2k(k+1)}{\sinh^2(x_i - x_j)} + \frac{2k(k+1)}{\sinh^2(x_i + x_j)} \right) - \sum_{i=1}^{N} \left(\frac{p(p+2q+1)}{\sinh^2 x_i} + \frac{4q(q+1)}{\sinh^2 2x_i} \right),$$

depends on 3 parameters k, p, q.

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depends on 3 parameters k, p, q.

We define the BC_∞ CM operator as

$$\mathcal{L}^{(k,p,q,h)} = \sum_{a,b>0} (p_{a+b} + 2p_{a+b-1})\partial_a \partial_b - k \sum_{a=2}^{\infty} \left[\sum_{b=0}^{a-2} p_{a-b-1}(2p_b + p_{b+1}) \right] \partial_a$$

$$+ \sum_{a=1}^{\infty} \left[(a + k(a+1) + 2h)p_a + (2a - 1 + 2ka + 2h - p)p_{a-1} \right] \partial_a,$$

where

$$p_0 = -k^{-1}(h + \frac{1}{2}p + q).$$

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Its eigenfunctions $\mathcal{J}_\lambda(u; k, p, q, h)$ are called **Jacobi symmetric functions.**

Symmetries of BC_∞ CM operator

Consider the following automorphisms ω and θ of the algebra Λ :

$$\omega(p_i) = k^{-1} p_i,$$

$$\theta(p_i) = p_i + (-2)^i \frac{2k + 1 - 2q}{2k}.$$

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The BC_∞-operator $\mathcal{L}^{(k,p,q,h)}$ has the following symmetries (dualities):

$$\mathcal{L}^{(k,p,q,h)} = \mathcal{L}^{(k,p',q',h)},$$

$$\omega \circ \mathcal{L}^{(k,p,q,h)} = k\mathcal{L}^{(k^{-1},r,s,\hat{h})} \circ \omega,$$

$$\theta \circ \mathcal{L}^{(k,p,q,h)} = \mathcal{L}^{(k,\tilde{p},\tilde{q},h)} \circ \theta,$$

where

$$p' = 1 + 2k - p - 2q, \;\; q' = q,$$

$$p = kr, \;\; (2q+1) = k(2s+1), \;\; 2\hat{h} - 1 = k^{-1}(2h-1)$$

$$\tilde{p} = -p, \;\; \tilde{q} = 2k + 1 - q.$$

Theorem 2. *The following diagram is commutative for $h = -km - n - \frac{1}{2}p - q$:*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathcal{L}^{(k,p,q,h)}} & \Lambda \\ \downarrow \varphi_{m,n} & & \downarrow \varphi_{m,n} \\ \Lambda_{m,n,k} & \xrightarrow{\mathcal{L}^{BC}_{m,n}(k,p,q)} & \Lambda_{m,n,k} \end{array}$$

Deformed BC Calogero-Moser operators as restrictions

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In other words, the deformed CM operator $\mathcal{L}^{BC}_{m,n}(k,p,q)$ is a **restriction of** BC_∞ **CM operator** onto the corresponding subvariety $Spec\,\Lambda_{m,n,k}$.

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Corollary. *For classical series deformed Calogero-Moser problems are integrable.*

Back to Lie superalgebras: orthosymplectic Euler supercharacters

The image of Jacobi symmetric functions

$$\mathcal{S}J_\lambda(u, v; k, p, q) = \varphi_{m,n}(\mathcal{J}_\lambda(x; k, p, q, h)),$$

where $h = -km - n - \frac{1}{2}p - q$, are called **super Jacobi polynomials.** Their specialized version have an interesting interpretation in representation theory of orthosymplectic Lie superalgebras $\mathfrak{osp}(M, 2N)$.

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where $h = -km - n - \frac{1}{2}p - q$, are called **super Jacobi polynomials.** Their specialized version have an interesting interpretation in representation theory of orthosymplectic Lie superalgebras $\mathfrak{osp}(M, 2N)$.

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Theorem 3. *For a special choice of parabolic subgroup the Euler supercharacters of* $\mathfrak{osp}(2m+1, 2n)$ *coincide with specialized super Jacobi polynomials*

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A similar fact holds for Lie superalgebra $\mathfrak{osp}(2m, 2n)$ and $\mathcal{S}J_\lambda(u, v; -1, 0, 0)$

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Corollary: **Pieri and Jacobi-Trudy formulas** for Euler supercharacters

Some related stuff

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