1D Supersymmetry and Oxidation

Francesco Toppan

CBPF (TEO), Rio de Janeiro (RJ), Brazil

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Based on:
M. Gonzales, S. Khodaee, F.T.
arXiv:1006.4678.

L. Baulieu, F.T. In preparation

- Irreps and indecomposable reducible reps.
- Oxidation.
- Non-minimal linear reps. of the $\mathcal{N} = 4$ Extended Susy in 1D.
- Susy Extension of Schur lemma.
- 1D Supersymmetric Sigma models.
- Twist supersymmetry.

Previous works on $1D$ susy reps:

- A. Pashnev and F. Toppan, *J. Math. Phys.* **42** (2001), 5257 (hep-th/0010135).
- M. Faux and S. J. Gates Jr., *Phys. Rev.* **D 71** (2005), 065002 (hep-th/0408004).
- Z. Kuznetsova, M. Rojas and F. Toppan, *JHEP* **0603** (2006), 098 (hep-th/0511274).
- C.F. Doran, M. G. Faux, S. J. Gates Jr., T. Hubsch, K. M. Iga and G. D. Landweber, math-ph/0603012.
- C.F. Doran, M. G. Faux, S. J. Gates Jr., T. Hubsch, K. M. Iga and G. D. Landweber, hep-th/0611060.
- F. Toppan, *POS* **IC2006**, 033 (hep-th/0610180).
- F. Toppan, in *Quantum, Super and Twistors*, Proc. 22nd Max Born Symp., Wrocław 2006.
Eds. Kowalski-Glikman and Turko (2008), 143 (hep-th/0612276).
- Z. Kuznetsova and F. Toppan, *Mod. Phys. Lett.* **A 23** (2008), 37 (hep-th/0701225).
- Z. Kuznetsova and F. Toppan, *Int. J. Mod. Phys.* **A 23** (2008), 3947 (arXiv:0712.3176).
- F. Toppan, *Acta Polyt.* **48** (2008), 56.
- C.F. Doran, M. G. Faux, S. J. Gates Jr., T. Hubsch, K. M. Iga, G. D. Landweber and R. L. Miller, arXiv:08060050.
- C.F. Doran, M. G. Faux, S. J. Gates Jr., T. Hubsch, K. M. Iga and G. D. Landweber, arXiv:08060051.

The $1D$ $\mathcal{N}$-Extended Superalgebra, with $\mathcal{N}$ odd generators Q_I $(I = 1, 2, \ldots, \mathcal{N})$ and a single even generator H satisfying the (anti)-commutation relations

$$\begin{aligned} \{Q_I, Q_J\} &= \delta_{IJ} H, \\ [H, Q_I] &= 0, \end{aligned}$$

The *minimal* linear representations (also called *irreducible supermultiplets*) are given by the minimal number n_{min} of bosonic (fermionic) fields for a given value of $\mathcal{N}$.

$$\begin{aligned} \mathcal{N} &= 8l + m, \\ n_{min} &= 2^{4l} G(m), \end{aligned}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

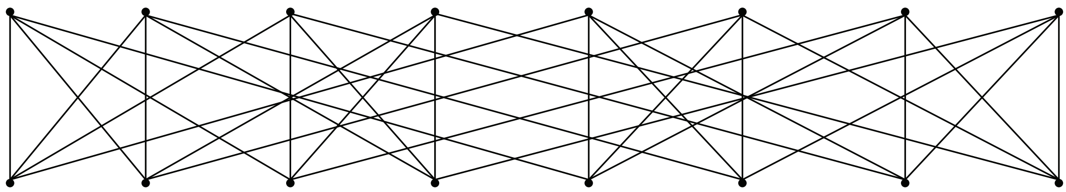
$G(m)$ appearing in (1) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

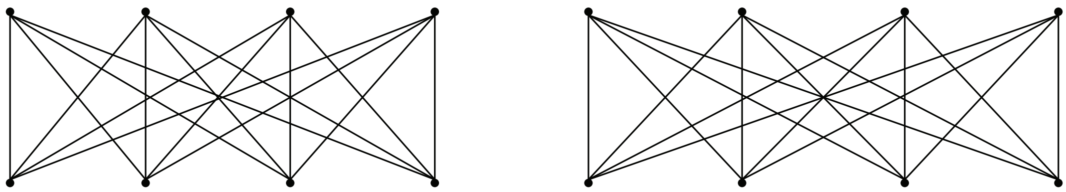
The maximal *finite* number n_{max} of bosonic (fermionic) fields entering a non-minimal represesentation

$$n_{max} = 2^{\mathcal{N}-1}.$$

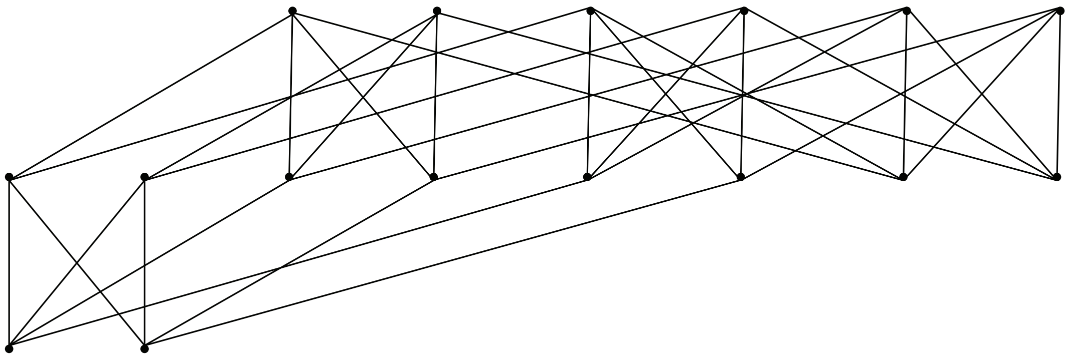
Graphical presentation of the $1D$ SUSY irreps (unoriented, color-blind graphs):



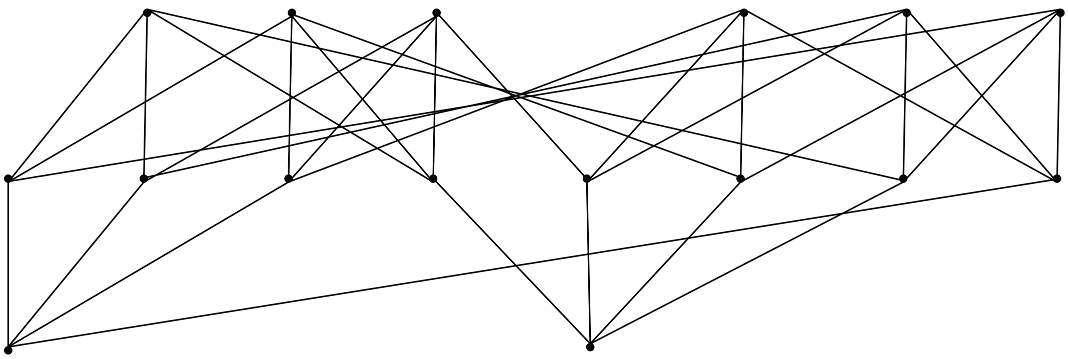
$(8,8)_{red}$, reducible but indecomposable.



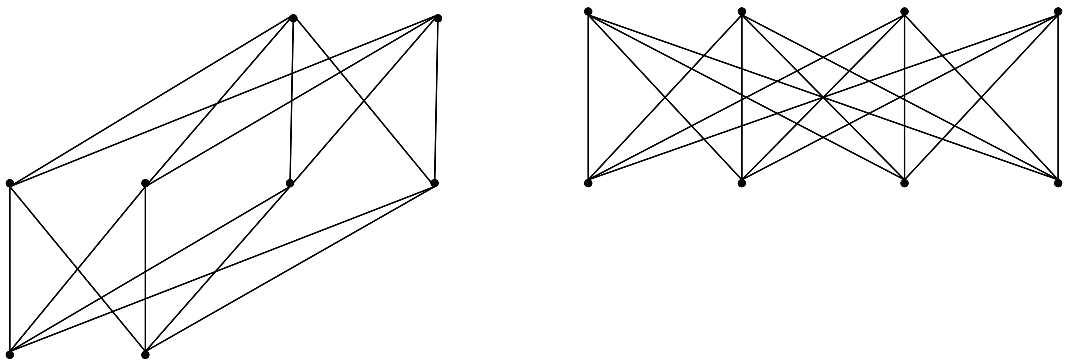
$(8,8)_{FR}$, fully reducible.



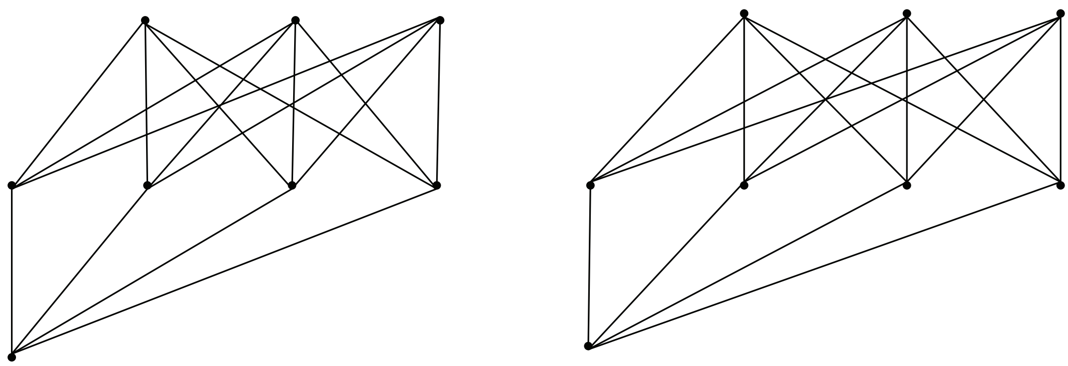
$(2,8,6)_A$, reducible but indecomposable with connectivity symbol $2_4 + 4_3 + 2_2$.



$(2,8,6)_B$, reducible but indecomposable with connectivity symbol 8_3.



$(2,8,6)_a$, fully reducible with connectivity symbol $4_4 + 4_2$.



$(2,8,6)_b$, fully reducible with connectivity symbol 8_3.

Non-minimal, reducible but indecomposable $\mathcal{N} = 4$ supermultiplets of field content $(k, 8, 8-k)$:

field content:	*label:*	*connectivity symbol:*	*dual supermultiplet:*
$(1, 8, 7)$		$4_4 + 4_3$	$(7, 8, 1)$
$(2, 8, 6)$	A	$2_4 + 4_3 + 2_2$	$(6, 8, 2)_A$
	B	8_3	$(6, 8, 2)_B$
$(3, 8, 5)$	A	$1_4 + 3_3 + 3_2 + 1_1$	$(5, 8, 3)_A$
	B	$4_3 + 4_2$	$(5, 8, 3)_B$
$(4, 8, 4)$	A	$1_4 + 6_2 + 1_0$	self-dual
	B	$4_3 + 4_1$	self-dual
	C	$2_3 + 4_2 + 2_1$	self-dual
	D	8_2	self-dual
$(5, 8, 3)$	A	$1_3 + 3_2 + 3_1 + 1_0$	$(3, 8, 5)_A$
	B	$4_2 + 4_1$	$(3, 8, 5)_B$
$(6, 8, 2)$	A	$2_2 + 4_1 + 2_0$	$(2, 8, 6)_A$
	B	8_1	$(2, 8, 6)_B$
$(7, 8, 1)$		$4_1 + 4_0$	$(1, 8, 7)$

$(\mathcal{N} = 4) \Rightarrow (\mathcal{N} = 5)$ oxidation:

$(2, 8, 6)$:

	$(\mathcal{N} = 4) \Rightarrow (\mathcal{N} = 5)$:	$2_5 + 2_4 + 4_3$	$6_4 + 2_3$
Connected:	$2_4 + 4_3 + 2_2$	X	X
	8_3		X
Disconnected:	$4_4 + 4_2$	X	
	8_3		X

$(3, 8, 5)$:

	$(\mathcal{N} = 4) \Rightarrow (\mathcal{N} = 5)$:	$1_5 + 3_4 + 4_2$	$2_4 + 5_3 + 1_2$
Connected:	$1_4 + 3_3 + 3_2 + 1_1$	X	X
	$4_3 + 4_2$		X
Disconnected:	$4_4 + 4_1$	X	
	$4_3 + 4_2$		X

$(4, 8, 4)$:

	$(\mathcal{N} = 4) \Rightarrow (\mathcal{N} = 5)$:	$4_4 + 4_1$	$1_4 + 3_3 + 3_2 + 1_1$	$4_3 + 4_2$
Conn.:	$1_4 + 6_2 + 1_0$		X	
	$4_3 + 4_1$	X		X
	$2_3 + 4_2 + 2_1$		X	X
	8_2			X
Disconn.:	$4_4 + 4_0$	X		
	$4_3 + 4_1$		X	
	8_2			X

$(5, 8, 3)$:

	$(\mathcal{N} = 4) \Rightarrow (\mathcal{N} = 5)$:	$4_3 + 3_1 + 1_0$	$1_3 + 5_2 + 2_1$
Connected:	$1_5 + 3_2 + 3_1 + 1_0$	X	X
	$4_2 + 4_1$		X
Disconnected:	$4_3 + 4_0$	X	
	$4_2 + 4_1$		X

$(6, 8, 2)$:

	$(\mathcal{N} = 4) \Rightarrow (\mathcal{N} = 5)$:	$4_2 + 2_1 + 2_0$	$2_2 + 6_1$
Connected:	$2_2 + 4_1 + 2_0$	X	X
	8_1		X
Disconnected:	$4_2 + 4_0$	X	
	8_1		X

Non-minimal $\mathcal{N} = 4$ supermultiplets and invariant groups (extension of Schur's lemma):

supermultiplet:	*commuting group:*
$(2,8,6)_A$	$U(1)$
$(2,8,6)_B$	$\mathbf{1}$
$(4,8,4)_A$	$\mathbf{1}$
$(4,8,4)_B$	$SU(2)$
$(4,8,4)_C$	$\mathbf{1}$
$(4,8,4)_D$	$U(1) \otimes U(1)$
$(6,8,2)_A$	$U(1)$
$(6,8,2)_B$	$\mathbf{1}$

In the remaining cases, for field content $(k, 8, 8-k)$ with k odd, the most general unitary group is just the identity group $\mathbf{1}$.

Sigma-models: two constructions.

I) Manifest construction:

$$\mathcal{S} \;=\; \int dt\mathcal{L} = \frac{1}{m}\int dt Q_1Q_2Q_3Q_4F(\vec{x})$$

F is the (unconstrained) prepotential.

II) Constrained prepotential: from the root supermultiplet, take $\overline{\mathcal{L}}$ and impose, for $j = k + 1,\ldots,8$,

$$\frac{\partial\overline{\mathcal{L}}}{\partial x_j} \;=\; 0,$$

eliminating the dependence on x_j's. This condition allows us to regard, according to the dressing procedure, the $\dot{x}_j$'s no longer as derivative fields, but as the auxiliary fields g_j of mass-dimension 1 entering the $(k, 8, 8-k)$ supermultiplet. We can therefore set

$$\begin{aligned} g_j &= \dot{x}_j \\ \overline{\mathcal{L}} &\equiv \overline{\mathcal{L}}(x_l, \dot{x}_l, \psi_i, \dot{\psi}_i, g_j), \end{aligned}$$

($l = 1,\ldots,k$, while $i = 1,\ldots,8$ and $j = k + 1,\ldots,8$).

Construction I: $\mathcal{N}=4$ off-shell invariant actions produces a first-order Lagrangian for

$$(1,8,7)_{red}, (2,8,6)_A, (3,8,5)_A, (4,8,4)_A, (4,8,4)_B.$$

With the only exception of $(4,8,4)_B$, these are the supermultiplets admitting fermionic sources.

In the remaining cases,

$$(2,8,6)_B, (3,8,5)_B, (4,8,4)_C, (4,8,4)_D, (5,8,3)_A,$$
$$(5,8,3)_B, (6,8,2)_A, (6,8,2)_B, (7,8,1)_{red},$$

the Construction I produces a second-order Lagrangian.

Example of manifest $\mathcal{N} = 4$ off-shell action for $(4, 8, 4)_C$:

$$
\begin{aligned}
\mathcal{L} \;=\; & \Phi(\dot\upsilon_1^2 + \dot\upsilon_2^2 + \dot\upsilon_3^2 + \dot{\bar\upsilon}_1^2 + g_0^2 + \bar g_0^2 + \bar g_2^2 + \bar g_3^2 + \dot\lambda_0\lambda_0 + \dot{\bar\lambda}_0\bar\lambda_0 + \\
& \dot\lambda_1\lambda_1 + \dot\lambda_2\lambda_2 + \dot\lambda_3\lambda_3 + \dot{\bar\lambda}_1\bar\lambda_1 + \dot{\bar\lambda}_2\bar\lambda_2 + \dot{\bar\lambda}_3\bar\lambda_3) + \\
& \Phi_1[\dot\upsilon_3(\bar\lambda_0\bar\lambda_2 + \lambda_2\lambda_0 + \bar\lambda_1\bar\lambda_3 + \lambda_1\lambda_3) - \dot\upsilon_2(\bar\lambda_0\bar\lambda_3 + \lambda_3\lambda_0 - \bar\lambda_1\bar\lambda_2 - \lambda_1\lambda_2) + \\
& + g_0(\bar\lambda_2\bar\lambda_3 - \lambda_2\lambda_3 - \bar\lambda_0\bar\lambda_1 + \lambda_1\lambda_0) + \bar g_0(\lambda_0\bar\lambda_1 + \lambda_1\bar\lambda_0 + \lambda_2\bar\lambda_3 - \lambda_3\bar\lambda_2) + \\
& \bar g_2(\lambda_2\bar\lambda_1 + \lambda_1\bar\lambda_2 - \lambda_0\bar\lambda_3 + \lambda_3\bar\lambda_0) + \bar g_3(\lambda_0\bar\lambda_2 - \lambda_2\bar\lambda_0 + \lambda_3\bar\lambda_1 + \lambda_1\bar\lambda_3)] + \\
& \Phi_2[\dot\upsilon_3(\bar\lambda_0\bar\lambda_1 + \lambda_1\lambda_0 + \bar\lambda_2\bar\lambda_3 + \lambda_2\lambda_3) + \dot\upsilon_1(\bar\lambda_0\bar\lambda_3 + \lambda_3\lambda_0 - \bar\lambda_1\bar\lambda_2 - \lambda_1\lambda_2) + \\
& + g_0(\bar\lambda_3\bar\lambda_1 - \lambda_3\lambda_1 - \bar\lambda_0\bar\lambda_2 + \lambda_2\lambda_0) + \bar g_0(\lambda_0\bar\lambda_2 + \lambda_2\bar\lambda_0 + \lambda_2\bar\lambda_3 - \lambda_3\bar\lambda_2) + \\
& - \bar g_2(\lambda_0\bar\lambda_0 + \lambda_1\bar\lambda_1 - \lambda_2\bar\lambda_2 + \lambda_3\bar\lambda_3) + \bar g_3(\lambda_3\bar\lambda_2 + \lambda_2\bar\lambda_3 - \lambda_0\bar\lambda_1 + \lambda_1\bar\lambda_0)] + \\
& \Phi_3[\dot\upsilon_2(\bar\lambda_0\bar\lambda_1 + \lambda_1\lambda_0 - \bar\lambda_2\bar\lambda_3 - \lambda_2\lambda_3) - \dot\upsilon_1(\bar\lambda_0\bar\lambda_2 + \lambda_2\lambda_0 + \bar\lambda_1\bar\lambda_3 + \lambda_1\lambda_3) + \\
& + g_0(\bar\lambda_1\bar\lambda_2 - \lambda_1\lambda_2 - \bar\lambda_0\bar\lambda_3 + \lambda_3\lambda_0) + \bar g_0(\lambda_0\bar\lambda_3 + \lambda_3\bar\lambda_0 + \lambda_1\bar\lambda_2 - \lambda_2\bar\lambda_1) + \\
& \bar g_2(\lambda_0\bar\lambda_1 - \lambda_1\bar\lambda_0 + \lambda_2\bar\lambda_3 + \\
& \lambda_3\bar\lambda_2) - \bar g_3(\lambda_0\bar\lambda_0 + \lambda_1\bar\lambda_1 + \lambda_2\bar\lambda_2 - \lambda_3\bar\lambda_3)] + \\
& \Phi_{\bar 1}[\dot\upsilon_1(\lambda_0\bar\lambda_0 - \lambda_1\bar\lambda_1 + \lambda_2\bar\lambda_2 + \lambda_3\bar\lambda_3) - \\
& \dot\upsilon_2(\lambda_0\bar\lambda_3 - \lambda_3\bar\lambda_0 + \lambda_2\bar\lambda_1 + \lambda_1\bar\lambda_2) + \\
& \dot\upsilon_3(\lambda_0\bar\lambda_2 - \lambda_2\bar\lambda_0 - \lambda_3\bar\lambda_1 - \lambda_1\bar\lambda_3) + g_0(\lambda_2\bar\lambda_3 - \lambda_3\bar\lambda_2 - \lambda_0\bar\lambda_1 - \lambda_1\bar\lambda_0) + \\
& \bar g_0(-\bar\lambda_0\bar\lambda_1 + \lambda_1\lambda_0 - \bar\lambda_2\bar\lambda_3 + \lambda_2\lambda_3) + \bar g_2(\bar\lambda_0\bar\lambda_3 + \lambda_3\lambda_0 + \bar\lambda_2\bar\lambda_1 + \lambda_2\lambda_1) + \\
& \bar g_3(-\bar\lambda_0\bar\lambda_2 - \lambda_2\lambda_0 + \bar\lambda_3\bar\lambda_1 + \lambda_3\lambda_1)] + \\
& \Phi_{\bar 1\bar 1}(\bar\lambda_1\lambda_1\lambda_2\bar\lambda_2 + \bar\lambda_1\lambda_1\lambda_3\bar\lambda_3 + \bar\lambda_0\lambda_1 + \\
& \bar\lambda_2\lambda_3 + \bar\lambda_0\lambda_2\bar\lambda_3\lambda_1 + \bar\lambda_0\lambda_1\bar\lambda_2\lambda_3 + \\
& \bar\lambda_0\lambda_3\bar\lambda_1\lambda_2 - \bar\lambda_1\bar\lambda_2\bar\lambda_3\bar\lambda_0) + \\
& \Phi_{11}(\bar\lambda_0\lambda_0\lambda_1\bar\lambda_1 + \bar\lambda_0\lambda_2\lambda_3\bar\lambda_1 + \bar\lambda_1\lambda_1\lambda_2\bar\lambda_2 + \\
& \bar\lambda_1\lambda_1\lambda_3\bar\lambda_3 + \lambda_0\lambda_2\bar\lambda_3\bar\lambda_1 + \lambda_0\lambda_3\bar\lambda_1\bar\lambda_2 \\
& - \lambda_1\lambda_2\lambda_3\lambda_0) + \\
& \Phi_{22}(\bar\lambda_0\lambda_0\lambda_2\bar\lambda_2 + \bar\lambda_0\lambda_3\lambda_1\bar\lambda_2 + \bar\lambda_2\lambda_2\lambda_1\bar\lambda_1 + \\
& \bar\lambda_2\lambda_2\lambda_3\bar\lambda_3 + \lambda_0\lambda_1\bar\lambda_2\bar\lambda_3 + \lambda_0\lambda_3\bar\lambda_1\bar\lambda_2 - \\
& - \lambda_1\lambda_2\lambda_3\lambda_0) + \\
& \Phi_{33}(\bar\lambda_0\lambda_0\lambda_3\bar\lambda_3 + \bar\lambda_0\lambda_1\lambda_2\bar\lambda_3 + \\
& \bar\lambda_3\lambda_3\lambda_1\bar\lambda_1 + \bar\lambda_3\lambda_3\lambda_2\bar\lambda_2 + \lambda_0\lambda_1\bar\lambda_2\bar\lambda_3 + \lambda_0\lambda_2\bar\lambda_3\bar\lambda_1 \\
& - \lambda_1\lambda_2\lambda_3\lambda_0) + \\
& \Phi_{\bar 1 1}(\lambda_1\lambda_0\lambda_2\bar\lambda_3 - \bar\lambda_0\bar\lambda_1\bar\lambda_2\lambda_3 + \\
& \lambda_0\lambda_3\lambda_1\bar\lambda_2 + \bar\lambda_0\bar\lambda_3\bar\lambda_1\lambda_2 + \lambda_0\lambda_2\lambda_3\bar\lambda_1 - \bar\lambda_0\bar\lambda_2\bar\lambda_3\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
&-\lambda_1\lambda_2\lambda_3\bar{\lambda}_0 - \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\lambda_0) + \\
&\Phi_{\bar{1}2}(\lambda_0\lambda_2\lambda_3\bar{\lambda}_2 + \bar{\lambda}_2\bar{\lambda}_1\lambda_0\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_2 + \\
&\lambda_0\lambda_3\lambda_1\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_3\bar{\lambda}_1\lambda_1 - \lambda_1\lambda_2\lambda_3\bar{\lambda}_3 \\
&-\bar{\lambda}_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3) + \\
&\Phi_{\bar{1}3}(\lambda_0\lambda_2\lambda_3\bar{\lambda}_3 + \bar{\lambda}_3\bar{\lambda}_1\lambda_0\bar{\lambda}_0 + \\
&\bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_3 + \lambda_0\lambda_1\lambda_2\bar{\lambda}_1 - \\
&\bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\lambda_1 - \lambda_1\lambda_3\lambda_2\bar{\lambda}_2 \\
&-\bar{\lambda}_1\bar{\lambda}_3\lambda_2\bar{\lambda}_2) + \\
&\Phi_{12}(\bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_2 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_2 - \\
&\lambda_0\bar{\lambda}_2\bar{\lambda}_3\lambda_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_1 - \lambda_0\bar{\lambda}_3\bar{\lambda}_1\lambda_1 \\
&+\bar{\lambda}_1\lambda_2\lambda_3\bar{\lambda}_3 + \bar{\lambda}_2\lambda_1\lambda_3\bar{\lambda}_3) + \\
&\Phi_{13}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_1 - \\
&\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_1 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_3 - \lambda_0\bar{\lambda}_2\bar{\lambda}_3\lambda_3 \\
&+\bar{\lambda}_1\lambda_3\lambda_2\bar{\lambda}_2 + \bar{\lambda}_3\lambda_1\lambda_2\bar{\lambda}_2) + \\
&\Phi_{23}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_2 - \lambda_0\bar{\lambda}_0\lambda_2\bar{\lambda}_3 + \\
&\bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_2 - \lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_3 - \lambda_0\bar{\lambda}_3\bar{\lambda}_1\lambda_3 \\
&+\bar{\lambda}_2\lambda_3\lambda_1\bar{\lambda}_1 + \bar{\lambda}_3\lambda_2\lambda_1\bar{\lambda}_1) + \\
&\Omega(\dot{\upsilon}_1\dot{\bar{\upsilon}}_1 + g_0\bar{g}_0 + \dot{\upsilon}_2\bar{g}_2 + \\
&\dot{\upsilon}_3\bar{g}_3 + \lambda_0\dot{\bar{\lambda}}_0 + \lambda_1\dot{\bar{\lambda}}_1 + \lambda_2\dot{\bar{\lambda}}_2 + \lambda_3\dot{\bar{\lambda}}_3) + \\
&\Omega_1(\dot{\upsilon}_1\lambda_1\bar{\lambda}_1 + \dot{\upsilon}_2\lambda_2\bar{\lambda}_1 + \dot{\upsilon}_3\lambda_3\bar{\lambda}_1 + \\
&g_0\lambda_0\bar{\lambda}_1 - \bar{g}_0\lambda_2\lambda_3 + \bar{g}_2\lambda_0\lambda_3 - \bar{g}_3\lambda_0\lambda_2) + \\
&\Omega_2(\dot{\upsilon}_1\lambda_1\bar{\lambda}_2 + \dot{\upsilon}_2\lambda_2\bar{\lambda}_2 + \dot{\upsilon}_3\lambda_3\bar{\lambda}_2 - \dot{\bar{\upsilon}}_1\lambda_0\lambda_3 - \\
&g_0\bar{\lambda}_2\lambda_0 - \bar{g}_0\lambda_3\lambda_1 + \bar{g}_3\lambda_0\lambda_1) + \\
&\Omega_3(\dot{\upsilon}_1\lambda_1\bar{\lambda}_3 + \dot{\upsilon}_2\lambda_2\bar{\lambda}_3 + \\
&\dot{\upsilon}_3\lambda_3\bar{\lambda}_3 + \dot{\bar{\upsilon}}_1\lambda_0\lambda_2 - g_0\bar{\lambda}_3\lambda_0 - \bar{g}_0\lambda_1\lambda_2 - \bar{g}_2\lambda_0\lambda_1) + \\
&\Omega_{\bar{1}}(\dot{\upsilon}_1(\lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3) + \\
&\dot{\upsilon}_2\bar{\lambda}_0\bar{\lambda}_3 - \dot{\upsilon}_3\bar{\lambda}_0\bar{\lambda}_2 - g_0\bar{\lambda}_2\bar{\lambda}_3 + \bar{g}_0\bar{\lambda}_0\lambda_1 + \bar{g}_2\bar{\lambda}_2\lambda_1 + \bar{g}_3\bar{\lambda}_3\lambda_1) + \\
&\Omega_{\bar{1}1}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_1 + \lambda_1\bar{\lambda}_1\lambda_2\bar{\lambda}_2 + \lambda_1\bar{\lambda}_1\lambda_3\bar{\lambda}_3) + \\
&\Omega_{\bar{1}2}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_2 + \lambda_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3) + \\
&\Omega_{\bar{1}3}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_3 + \lambda_1\bar{\lambda}_3\lambda_2\bar{\lambda}_2) + \\
&\Omega_{12}(\lambda_2\lambda_0\lambda_3\bar{\lambda}_2 - \lambda_0\lambda_3\lambda_1\bar{\lambda}_1) + \\
&\Omega_{13}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_1 - \lambda_0\lambda_2\lambda_3\bar{\lambda}_3) + \\
&\Omega_{23}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_2 - \lambda_0\lambda_3\lambda_1\bar{\lambda}_3),
\end{aligned}
$$

where $F(\upsilon_1,\upsilon_2,\upsilon_3,\bar{\upsilon}_1)$ is the unconstrained prepotential and

$$\begin{aligned} \Omega &= \nabla F = \partial_{11}F + \partial_{22}F + \partial_{33}F + \partial_{\bar{1}\bar{1}}F, \\ \Phi &= \partial_{1\bar{1}}F. \end{aligned}$$

The constraints $\nabla\Phi = 0$ and $\Omega = 0$ arise as a consequence of imposing an extra invariance under an $\mathcal{N} = 5$-Extended Supersymmetry (under such constraints the resulting off-shell action is also automatically $\mathcal{N} = 8$-invariant).

Explicit example of $\mathcal{N} = 4$-invariant, first-order action derived from Construction I:

$(4,8,4)_B$ non-minimal supermultiplet with connectivity symbol $4_3 + 4_1$ and component fields $(\upsilon_0, \upsilon_i; \lambda_0, \lambda_i, \bar{\lambda}_0, \bar{\lambda}_i; \bar{g}_0, \bar{g}_i)$ $(i = 1,2,3)$:

$$
\begin{aligned}
\mathcal{L} \;=\;& \Omega(\dot{v}_0\bar{g}_0 + \dot{v}_i\bar{g}_i) + \\
& \Omega(\lambda_0\dot{\bar{\lambda}}_0 + \lambda_k\dot{\bar{\lambda}}_k) + \\
& \varepsilon_{ijk}\Omega_j\bar{g}_k\lambda_0\lambda_i - \\
& \Omega_i\dot{v}_0\bar{\lambda}_i\lambda_0 - \Omega_0\dot{v}_i\bar{\lambda}_0\lambda_i + \\
& \frac{\varepsilon_{ijk}}{2}(\Omega_0\bar{g}_k - \Omega_k\bar{g}_0)\lambda_i\lambda_j - \\
& \Omega_j\dot{v}_i\bar{\lambda}_j\lambda_i - \Omega_0\dot{v}_0\bar{\lambda}_0\lambda_0 \\
& -\frac{1}{2}\varepsilon_{ijk}\Omega_{0k}\lambda_i\lambda_j\lambda_0\bar{\lambda}_0 - \\
& \frac{\varepsilon_{ijk}}{2}\delta_{pq}\Omega_{pk}\lambda_0\lambda_i\lambda_j\bar{\lambda}_q \\
& -\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0 - \Omega_{0p}\lambda_p)\lambda_i\lambda_j\lambda_k,
\end{aligned}
$$

where

$$
\Omega \;=\; \partial_{00}F + \partial_{ii}F.
$$

Construction II on $(4,8,4)_B$.

The $\mathcal{N}=4$-invariance is recovered iff the constraint

$$\Phi_{00}+\Phi_{ii} \;=\; 0$$

is satisfied.

$$
\begin{aligned}
\mathcal{L} \;=\;& \Phi(\dot{v}_0^2+\dot{v}_i^2+\bar{g}_0^2+\bar{g}_i^2+\\
&\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\lambda}_i\lambda_i+\dot{\bar{\lambda}}_i\bar{\lambda}_i)+\\
&\varepsilon_{ijk}\Phi_k\dot{v}_j(\bar{\lambda}_0\bar{\lambda}_i+\lambda_i\lambda_0)+(\Phi_0\dot{v}_i-\Phi_i\dot{v}_0)(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)\\
&-\varepsilon_{ijk}\Phi_j\bar{g}_k(\lambda_0\bar{\lambda}_i-\lambda_i\bar{\lambda}_0)+(\Phi_0\bar{g}_i+\Phi_i\bar{g}_0)(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)+\\
&\frac{\varepsilon_{ijk}}{2}(\Phi_i\dot{v}_0-\Phi_0\dot{v}_i)(\bar{\lambda}_j\bar{\lambda}_k-\lambda_j\lambda_k)+\\
&\frac{1}{2}(\Phi_j\dot{v}_k-\Phi_k\dot{v}_j)(\bar{\lambda}_j\bar{\lambda}_k+\lambda_j\lambda_k)+\\
&(\Phi_0\bar{g}_0+\Phi_j\bar{g}_j)(\bar{\lambda}_0\lambda_0+\\
&\bar{\lambda}_k\lambda_k)-\Phi_i\bar{g}_j(\bar{\lambda}_i\lambda_j-\lambda_i\bar{\lambda}_j)-\Phi_0\bar{g}_0(\bar{\lambda}_0\lambda_0-\lambda_0\bar{\lambda}_0)+\\
&\varepsilon_{ijk}\lambda_i\bar{\lambda}_j(\Phi_k\bar{g}_0-\Phi_0\bar{g}_k)+(\varepsilon_{ijk}\Phi_{0k}-\Phi_{ij})\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_j+\\
&\Phi_{0j}(\bar{\lambda}_0\lambda_j-\lambda_0\bar{\lambda}_j)\lambda_i\bar{\lambda}_i+\\
&\frac{1}{2}\varepsilon_{ijk}\delta_{pq}\Phi_{kp}(\bar{\lambda}_0\lambda_i\lambda_j\bar{\lambda}_q-\lambda_0\bar{\lambda}_i\bar{\lambda}_j\lambda_q)+\\
&\Phi_{jk}\bar{\lambda}_j\lambda_k\lambda_i\bar{\lambda}_i-\frac{1}{2}\varepsilon_{ijk}\delta_{pq}\Phi_{0p}\lambda_i\bar{\lambda}_j\lambda_k\bar{\lambda}_q\\
&-\Phi_{00}\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i+\\
&\frac{\varepsilon_{ijk}}{2}[\Phi_{pp}(\lambda_0\lambda_i\bar{\lambda}_j\bar{\lambda}_k)+\Phi_{00}(\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k)]+\\
&\Omega(\dot{v}_0\bar{g}_0+\dot{v}_i\bar{g}_i)+\Omega(\lambda_0\dot{\bar{\lambda}}_0+\lambda_k\dot{\bar{\lambda}}_k)+\\
&\varepsilon_{ijk}\Omega_j\bar{g}_k\lambda_0\lambda_i-\Omega_i\dot{v}_0\bar{\lambda}_i\lambda_0-\Omega_0\dot{v}_i\bar{\lambda}_0\lambda_i+\\
&\frac{\varepsilon_{ijk}}{2}(\Omega_0\bar{g}_k-\Omega_k\bar{g}_0)\lambda_i\lambda_j-\Omega_j\dot{v}_i\bar{\lambda}_j\lambda_i-\Omega_0\dot{v}_0\bar{\lambda}_0\lambda_0\\
&-\frac{1}{2}\varepsilon_{ijk}\Omega_{0k}\lambda_i\lambda_j\lambda_0\bar{\lambda}_0-\frac{\varepsilon_{ijk}}{2}\delta_{pq}\Omega_{pk}\lambda_0\lambda_i\lambda_j\bar{\lambda}_q\\
&-\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0-\Omega_{0p}\lambda_p)\lambda_i\lambda_j\lambda_k
\end{aligned}
$$

Complete classification of $\mathcal{N}=4$ irreps and reducible but indecomposable reps.
Application of reducible but indecomposable reps to partial breaking of susies.

Comments on oxidation:

Reconstruction of higher-dimensional supersymmetric theories from $1D$ supersymmetric data:
The reduction to a $0+1$ quantum mechanical system of the $\mathcal{N}=4$ SYM in $3+1$ dimension produces a supermultiplet of field content $(9,16,7)$ which carries an off-shell representation of 9 supercharges. The remaining 7 supersymmetry generators $(=16-9)$ can only be realized on-shell.

Similarly, for $11D$ SUGRA, the transverse coordinates supermultiplet containing the graviton, the gravitinos and the 3-form admits field content $(44,128,84)$.

It carries an off-shell representation for $\mathcal{N}=16$ supercharges.

The remaining 16 supersymmetry generators which complete the total number of $32 = 8 \times 4$ supercharges close on-shell. An off-shell formulation of the M-theory would require a supermultiplet with at least

$$32,768 = 2^{15}$$

bosonic component fields and an equal number of fermionic component fields.

The oxidation program can be carried out in two steps. In the first step an $\mathcal{N}$-extended, one-dimensional, supersymmetric theory has to be constructed for a sufficiently large value of $\mathcal{N}$. The most interesting values for $\mathcal{N}$ correspond to $\mathcal{N} = 4, 8, 16, 32$ which can be associated to a dimensional reduction of a 4, 6, 10, or 11-dimensional supersymmetric theory, respectively.

Twisted supersymmetry.

The $N=1$, $D=4$ vector and scalar multiplets and their twisted symmetries.
The $N=1$ vector and scalar multiplets are $(3,4,1)$ and $(2,4,2)$.
When the 4-manifold has holonomy $SU(2)$, the multiplets $(3,4,1)$ and $(2,4,2)$ can be represented as

$$A_m,\ A_{\overline{m}},\ \Psi_m,\ \chi_{\overline{mn}},\ \chi,\ h$$

and

$$\Phi,\ \bar{\Phi},\ \Psi_{\overline{m}},\ \chi_{mn},\ \bar{\chi},\ B_{\overline{mn}},\ B_{mn}.$$

Kaehler geometry: Kaehler metric g with $g_{m\overline{m}} = g_{\overline{m}m}$ and $g_{\overline{mn}} = g_{mn} = 0$ and a complex structure $J^i{}_j$ with $J^2 = 0$. Using the metric one can lower indices and define the Kaehler 2-form $J_{m\overline{n}} = -J_{\overline{n}m}$ and $J_{mn} = 0 = J_{\overline{mn}} = 0$

The twist as a change of variables: Dirac equivalent to

$$\bar{\lambda}\gamma^{\mu}D_{\mu}\lambda = \chi_{\overline{mn}}D_{[m}\Psi_{n]} + \chi D_{\overline{m}}\Psi_{m},$$

$$\begin{array}{rl}
A_m, & (gh. \quad n. = 0, mass \quad dim. = 0), \\
A_{\overline{m}} & (gh. \quad n. = 0, mass \quad dim. = 0), \\
\psi_m & (gh. \quad n. = +1, mass \quad dim. = \frac{1}{2}), \\
\chi_{\overline{mn}} & (gh. \quad n. = -1, mass \quad dim. = \frac{1}{2}), \\
\chi & (gh. \quad n. = -1, mass \quad dim. = \frac{1}{2}), \\
h & (gh. \quad n. = 0, mass \quad dim. = 1).
\end{array}$$

$$\begin{array}{rl}
\Phi, & (gh. \quad n. = 2, mass \quad dim. = 0), \\
\overline{\Phi} & (gh. \quad n. = -2, mass \quad dim. = 0), \\
\psi_{\overline{m}} & (gh. \quad n. = +1, mass \quad dim. = \frac{1}{2}), \\
\chi_{mn} & (gh. \quad n. = -1, mass \quad dim. = \frac{1}{2}), \\
\overline{\chi} & (gh. \quad n. = -1, mass \quad dim. = \frac{1}{2}), \\
T_{mn} & (gh. \quad n. = 0, mass \quad dim. = 1), \\
B_{mn} & (gh. \quad n. = 0, mass \quad dim. = 1).
\end{array}$$

TQFT symmetry:

$$\{s, s_{\overline{m}}\} = \partial_{\overline{m}} + (A_{\overline{m}}),$$
$$\{s_{\overline{p}}, s_{mn}\} = aJ_{\overline{p}[m}(\partial_{n]} + (A_{n]})$$

These operators anticommute with $d = \partial + \bar{\partial}$. The remianing anticommutators between $s, s_{\overline{p}}$ and s_{mn} are all vanishing (in particular they are nilpotent).

$$s \quad (gh. \quad n = 1, mass \quad dim. = 0),$$
$$s_{\overline{m}} \quad (gh.\ n. = -1, mass \quad dim. = 0),$$
$$s_{mn} \quad (gh. \quad n. = 1, mass \quad dim. = 0).$$

Gauge $A_{\overline{2}} = 0$:

$3D$ algebra $s, s_{\overline{p}}$ or $1D$ algebra $s, s_{\overline{p}}, s_{mn}$. Modulo gauge transformations, the transformations of the dimensionally reduced twisted vector multiplet $(3,4,1)$ are given by

	s	$s_{\overline{1}}$	$s_{\overline{2}}$	s_{12}
A_1	ψ_1	0	$-a\chi$	0
A_2	ψ_2	$a\chi$	0	0
$A_{\overline{1}}$	0	0	$-\chi_{\overline{12}}$	$a\psi_1$
ψ_1	0	0	$\dot{A}_1 + ah$	0
ψ_2	0	$-ah$	$\dot{A}_2$	0
$\chi_{\overline{12}}$	$-\dot{A}_{\overline{1}}$	0	0	$a(\dot{A}_1 + ah)$
χ	h	0	0	0
h	0	0	$\dot{\chi}$	0

$$
\begin{aligned}
z_1 &= A_1, \\
z_2 &= A_{\overline{1}} - aA_2, \\
z_3 &= A_{\overline{1}} + aA_2, \\
\xi_1 &= \psi_1 - a\chi, \\
\xi_2 &= \psi_1 + a\chi, \\
\xi_3 &= \chi_{\overline{12}}, \\
\xi_4 &= \chi_{\overline{12}} - a\psi_2, \\
g &= \dot{A}_1 + 2ah
\end{aligned}
$$

and introducing the basis of four operators

$$
\begin{array}{rcl}
s_{\pm} & = & s \pm s_{\overline{2}}, \\
N_{\pm} & = & \frac{1}{a}(s_{12} \pm s_{\overline{1}}),
\end{array}
$$

we obtain a realization of the $\mathcal{N} = (1, 1, 2)$ generalized supersymmetry. Indeed, the four operators $s_{\pm}, N_{\pm}$ are mutually anticommuting, while their square is

$$
{s_{\pm}}^2 = \pm\partial_t \quad , \quad {N_{\pm}}^2 = 0.
$$

The transformation table reads now as

	s_+	s_-	N_+	N_-
z_1	ξ_1	ξ_2	0	0
z_2	$-\xi_3$	ξ_4	ξ_1	ξ_2
z_3	$-\xi_4$	ξ_3	$-\xi_2$	ξ_1
ξ_1	$\dot{z}_1$	$-g$	0	0
ξ_2	g	$-\dot{z}_1$	0	0
ξ_3	$-\dot{z}_2$	$-\dot{z}_3$	$\dot{z}_1$	g
ξ_4	$-\dot{z}_3$	$-\dot{z}_2$	g	$\dot{z}_1$
g	$\dot{\xi}_2$	$\dot{\xi}_1$	0	0