ABELIAN HOPFIONS ON $\mathbb{R}^{2p+1}$: $p = 1$ and $p = 2$

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1 Topological charges

1.1 Chern–Pontryagin charges and their descendents

These are defined in all **even** dimensions, and dimensional descent enables their definition in all **odd** dimensions as well.

- *Chern–Pontryagin* charges stabilise instantons (Yang–Mills solitons) in all **even** dimensions. These are evaluated by surface integrals whose values depend only on the asymptotic properties of the fields. Famously, the asymptotic gauge connection behaves as a *pure gauge*.

  Instantons of suitable Yang–Mills models in all even $2p+2$ dimensions exist for suitable models defined in terms of $2p-$form curvature fields $F(2p) = F(2) \wedge F(2) ... \wedge F(2)$, and in particular in $4p$ saturate their topological lower bounds. However, for all $p \geq 1$ the selfduality equations are (progressively more) overdetermined and support nontrivial solutions only when subjected to suffiently high symmetry.

- *Descendents of Chern–Pontryagin* charges stabilise Yang–Mills–Higgs solitons on both **even** and **odd** $\mathbb{R}^D$, for the model described after dimensional reduction of the Yang–Mills system on $\mathbb{R}^D \times K^N$ $(D + K$ even) after integrating over the compact coset space K. One might call these "monopoles in all dimensions"

  - For **odd** D, the charge density is *gauge invariant*, *e.g.* monopole charge density
  - For **even** D, the charge density is *gauge variant*, *e.g.* vortex number density of the Abelian Higgs model
  - Bogomol'nyi bounds for monopoles in dimensions $D \geq 4$ are overdetermined and **cannot be saturated**
  - In contrast to instantons, the connection field of monopoles in all dimensions $D \geq 3$ behave as *one-half pure gauge*. This is a consequence of symmetry breaking asymptotics of the Higgs field. The resulting (slower) decay enables the definition of Dirac–Yang monopoles on the boundary
  - The topology of the monopole charge is **encoded in the Higgs field**, and is essentially a winding number as in Goldstone models

- *Winding numbers* for regular scalar fields, which map the configuration space on to the base space, stabilise the solitons of the corresponding models. Like the Pontryagin charge, they depend only on the asymptotic properties of the fields.

  - In the gauge decoupling limit of the above YMH models supporting monopoles (in all dimensions), we refer to the resulting scalar field model as a Goldstone model. These satsfy symmetry breaking asymptotics and their topological charge densities are the gauge decoupled limits of the monopole charge densities discussed in the above item. Like the latter, these charge densities are total divergences $\mapsto$ surface integral for the charge
  - For constrained, Sigma Model, fields by contrast the charge density is **not always** a total divergence, but is *essentially total divergence* in the sense that subjecting it to the variational principle results in **no equations of motion**. When a parametrisation of the fields incorporates

the sigma–model constraint, then these charge densities become total divergences.

- ∗ Sigma models featuring S^D valued fields on $\mathbb{R}^D$, *e.g.* the usual Skyrme model for $D = 3$, are as described above, but,
- ∗ Sigma models featuring complex valued fields, *e.g.* $\mathbb{CP}^n$ or Grassmannian valued fields are defined in terms of (in general) matrices z_1 and z_1

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \text{subject to} \qquad Z^\dagger Z = \mathbb{I} \tag{1}$$

(When the matrices z_1 and z_1 are replaced by $(n+1)-$component arrays these are the $\mathbb{CP}^n$ models.) Assuming that the models in question are invariant under the gauge transformations $Z \to Z\,g$, this enables the definition of a *composite connection*

$$B_i = Z^\dagger \partial_i Z\,. \tag{2}$$

with respect to which the covariant derivative

$$D_i Z = \partial_i Z - Z\,B_i \tag{3}$$

transforms as $D_i Z \to D_i Z\,g$.
The winding number densities in these models can be constructed as Pontryagin densities by directly using the curvatures G_{ij} of the composite connections (38). Clearly, these are by construction total divergence in contrast to the case of sigma models with S^D- fields.

The composite connections of the solitons of the sigma models are asymptotically *pure gauge*, like instantons.

In summary, solitons of the Goldstone models are akin to monopoles in that they both describe symmetry breaking solutions. By contrast solitons of the (complex) sigma models are akin to instantons in that their asymptotic composite gauge connections are pure gauge. This last property persists also for the gauged S^D sigma models, to be discussed immediately

- *Winding numbers of gauged scalar fields* stabilise the corresponding solitons of *a*) gauged Goldstone models and *b*) gauged Sigma models. With suitable gauging prescription and for appropriate choice of model, Bogomol'nyi bounds for both gauged Goldstone models and gauged Sigma models can be established . These bounds are never saturated.

Note: A gauged Goldstone model differs from a Yang–Mills–Higgs model resulting from the dimensional descent of Yang–Mills system. The topological lower bound of a gauged Goldstone model is always **higher** than that of the Yang–Mills–Higgs model featuring the same field multiplet.

1.2 Chern–Simons charges: Hopfions

The $O(3)$, or equivalently the $\mathbb{CP}^1$, sigma model is said to describe the ferromagnet. Belavin and Polyakov constructed all the solitons of this model, on $\mathbb{R}^2$. But the ferromagnet lives on $\mathbb{R}^3$ and not $\mathbb{R}^2$, so another type of soliton is needed for this purpose, *i.e., inter alia,* a new topological charge. This is the Hopf charge which is the volume integral of the Chern–Simons density constructed from the composite connection of the $\mathbb{CP}^1$ coordinate.

These charges stabilise solitons of (ungauged) Sigma Models. What distinguishes them from the usual Sigma Model solitons on $\mathbb{R}^D$ is that the scalar field multiplet involved is **not** that for which the usual soliton on $\mathbb{R}^D$ exists, but rather it is the scalar multiplet for which the usual soliton on $\mathbb{R}^{D-1}$ exists. As such one would not immediately expect the existence of solitons in this case since the definition of the usual winding number density is now lost because of the mismatch between the dimensions of the space and the scalar multiplet.

There exists however another global charge, which is topological in its origin in that its value depends exclusively on the asymptotic value of the field. This is the Chern–Simons charge which stabilises the Hopfion.

The following are the salient properties of the Chern–Simons densities:

- They are defined in **odd** spacelike dimensions **only**
- Unlike Chern–Pontryagin densities, the Chern–Simons (CS) densities **are not total divergences**. Hence it is obvious that the integral of the CS density for a Yang–Mills theory does not depend exclusively on the asymptotics and that it is not a topological charge. The definition of a CS density requires nevertheless the existence of a *connection* field
- To enable the definition of a globally defined topological charge that depends **only** on the asymptotic fields, the Chern–Simons density must be cast in

the form of an *essentially total divergence*. This is why the dynamics of the appropriate models must be that of a Sigma Model.

- To realise CS Hopfions therefore, it is convenient to employ complex projective sigma models for which one can readily define a composite connection as in (38)

- It is readily verified that the CS density of a complex Sigma Model described by Z, in (1), is indeed an *essentially total divergence* whose volume integral is a topological charge depending only on the asymptotic values Z takes. For example in the $D = 3$ case, subjecting the CS density

$$\Omega_{\mathrm{SU}(2)} = \frac{1}{2}\,\varepsilon_{ijk}\,\mathrm{Tr}\,B_k\left(G_{ij} - \frac{2}{3}\,B_i\,B_j\right) + \Lambda\,(\mathbb{1} - Z^\dagger\,Z)\,, \tag{4}$$

to the variational principle taking into account the constraint via the use of the Lagrange multiplier (matrix) Λ, yields

$$\varepsilon_{ijk}\,D_k Z\,G_{ij} = 0\,, \tag{5}$$

which is identically zero by virtue of the Bianchi identity.

Replacing the complex valued matrix Z with an array automatically gives the $\mathbb{CP}^n$ case. The formal extension of this demonstration to arbitrary $\mathbb{R}^{2p+1}$ proceeds systematically.

In the presentation at hand, we restrict to that Abelian case, namely to $\mathbb{CP}^n$ models on $\mathbb{R}^{2n+1}$, further restricting our detailed considerations to the well known case of $n = 1$, and to the case $n = 2$.

2 The CP^n models on $\mathbb{R}^{2n+1}$

The models are described by complex $n-$tuplets

$$Z = \begin{bmatrix} z^1 \\ z^2 \\ .. \\ .. \\ z^{n+1} \end{bmatrix} \equiv z^a \quad ; \quad a = 1, 2, ..., n+1\,, \tag{6}$$

subject to the constraint

$$Z^\dagger\,Z \equiv \bar z^a\,z^a = 1\,, \tag{7}$$

taking their values in $\frac{U(n+1)}{U(n)\times U(1)}$, such they are described by $2n+1$ real parameters that parametrise $\mathbb{R}^{2n+1}$.

The most interesting feature of these models is that when the field Z is subjected to a *local* $U(1)$ gauge transformation $g = \mathrm{e}^{i\Lambda(x)}$, then the quantity defined as

$$B_i = i\,Z^\dagger\partial_i Z \quad , \quad i = 1, 2, ..., 2n+1 \tag{8}$$

transforms like a *composite connection* under $g(\Lambda) = \mathrm{e}^{i\Lambda}$. This then enables the definitions of the curvature of this connection and the covariant derivative of Z with respect to it

$$G_{ij} \;=\; \partial_i\,B_j - \partial_j\,B_i \tag{9}$$

$$D_i Z \;=\; \partial_i\,Z + iB_i\,Z\,. \tag{10}$$

The Abelian CS density on $\mathbb{R}^{2n+1}$ is then readily defined in terms of the quantities (9) and (10). This is what makes these models well suited to describing Abelian Hopfions in all odd dimensions.

2.1 CP^1 models on $\mathbb{R}^3$

The most general [1] model supporting finite energy solutions, consistent with the Derrick scaling requirement is

$$\mathcal{H} = \kappa_0^0\,V + \frac{1}{2}\,\kappa_1^2\,D_i Z^\dagger\,D_i Z + \frac{1}{4}\,\kappa_2^4\,G_{ij}^2 \tag{11}$$

with $D_i Z$ and G_{ij} given by (10) and (9). The constants κ_0, κ_1, and κ_2 each have the dimension of length, and $V =$ is some pion mass type potential, which we will choose as

$$V = 1 + Z^\dagger\,\sigma_3\,Z\,. \tag{12}$$

In the special case with $\kappa_0 = 0$, (11) reduces to the Skyrme-Fadde'ev model.

[1] In the usual Skyrme model, namely the $O(4)$ model on $\mathbb{R}^3$, there is also a nonvanishing *sextic* term in the most general case. For the Skyrme-Fadde'ev model, namely the $\mathbb{CP}^1$ model on $\mathbb{R}^3$, this term vanishes since there are only *two* independent fields in this case in contrast with the *three* in the former.

The reduced 2 dimensional subsystem of (11) employing an Ansatz which gives a deformation of the field Z after it is subjected to axial symmetry in the (x_1, x_2) plane. The Anstaz used is

$$Z = \begin{bmatrix} a + ib \\ c\, e^{in\varphi} \end{bmatrix} \equiv \begin{bmatrix} \sin\frac{f}{2}\, e^{i\alpha} \\ \cos\frac{f}{2}\, e^{in\varphi} \end{bmatrix} \tag{13}$$

where the functions a, b, c, f and α all depend on both $\rho = \sqrt{|x_\alpha|^2}$ and $z \equiv x_3$, $\alpha = 1, 2$. The field (13) is not really axially symmetric, as long as $\alpha \neq 0$ (or $b \neq 0$), and as we shall see later the CS density vanishes unless $\alpha \neq 0$.

2.2 CP^2 models on $\mathbb{R}^5$

The most general [2] model supporting finite energy solutions, consistent with the Derrick scaling requirement is.

$$\mathcal{H} = \kappa_0^0\, V + \frac{1}{2}\,\kappa_1^2\, D_i Z^\dagger\, D_i Z + \frac{1}{4}\,\kappa_2^4\, G_{ij}^2 + \frac{1}{8}\,\kappa_3^6\,(G_{[ij}\, D_{k]} Z)^\dagger (G_{[ij}\, D_{k]} Z) + \frac{1}{16}\,\kappa_4^8\, G_{ijkl}^2 \tag{14}$$

with $D_i Z$ and G_{ij} given by (10) and (9), and the 4−form G_{ijkl} being the totally antisymmetrised product of this curvature. The constants κ_0, κ_1, $\kappa 2$, κ_3 and κ_4 each have the dimension of length, and V is some pion mass type potential. According to the scaling requirement for finite energy, it is necessary to retain *at least one* of the constants $(\kappa_0, \kappa_1, \kappa_2)$ and *at least one* of the constants (κ_3, κ_4), with the option of setting the rest equal to zero.

From the purely pragmatic viewpoint of algebraic manipulation the simplest model is

$$\mathcal{H}_{(2,4)} = \frac{1}{4}\,\kappa_2^4\, G_{ij}^2 + \frac{1}{16}\,\kappa_4^8\, G_{ijkl}^2 \tag{15}$$

2.2.1 Reduced CP^2 models on $\mathbb{R}^5$: via bi-azimuthal symmetry

The reduced 3 dimensional subsystem of (14) employing an Ansatz which gives a deformation of the field Z after it is subjected to two azimuthal (axial) symmetries in the (x_1, x_2) and (x_3, x_4) planes separately. The bi-azimuthally symmetric

[2] In the $O(6)$ sigma models on $\mathbb{R}^5$, there is also a nonvanishing *dectic* term $|G_{[ijkl}\, D_{m]} Z|^2$ in the most general case. In the $\mathbb{CP}^2$ model on $\mathbb{R}^5$, this term vanishes since there are only *four* independent fields in this case in contrast with the *five* in the former.

Ansatz here for the field (6), with $n=2$, is

$$Z=\left[\begin{array}{c} a(\rho,\sigma,z)+ib(\rho,\sigma,z) \\ c(\rho,\sigma,z)\,\mathrm{e}^{in\varphi} \\ d(\rho,\sigma,z)\,\mathrm{e}^{im\chi} \end{array}\right]\equiv\left[\begin{array}{c} \sin\frac{1}{2}f(\rho,\sigma,z)\,\mathrm{e}^{i\alpha(\rho,\sigma,z)} \\ \cos\frac{1}{2}f(\rho,\sigma,z)\,\sin g(\rho,\sigma,z)\,\mathrm{e}^{in\varphi} \\ \cos\frac{1}{2}f(\rho,\sigma,z)\,\cos g(\rho,\sigma,z)\,\mathrm{e}^{im\chi} \end{array}\right] \tag{16}$$

in terms of the variables $\rho=\sqrt{|x_\alpha|^2}$, $\sigma=\sqrt{|x_A|^2}$ with $\alpha=1,2,\ A=3,4$ and $z\equiv x_5$. φ and χ are the azimuthal angles in the (x_1,x_2) and (x_3,x_4) planes respectively, (n,m) being the winding (vortex) numbers of plane respectively. The field (16) is not really bi-azimuthally symmetric, as long as $\alpha\neq 0$ (or $b\neq 0$), and as we shall see later the CS density vanishes unless $\alpha\neq 0$.

3 Chern–Simons densities on $\mathbb{R}^3$ and on $\mathbb{R}^5$

3.1 Chern–Simons density on $\mathbb{R}^3$

The Chern–Simons density on $\mathbb{R}^3$, denoting the coordinates $x_i=(x_\mu,x_3)$, is

$$\begin{aligned}\Omega^{(3)}_{\mathrm{U}(1)} &= \varepsilon_{mij}\,B_m\,G_{ij} \\ &= \varepsilon_{\alpha\beta}\,(B_3\,G_{\alpha\beta}+2\,B_\alpha\,G_{\beta 3})\ .\end{aligned} \tag{17}$$

Substituting the azimuthally symmetric Ansatz (13) into (17) yields the simple expression

$$\frac{1}{2}\,\Omega^{(3)}_{\mathrm{CS}}=\det\left|\begin{array}{ccc} a & b & c \\ a_\rho & b_\rho & c_\rho \\ a_z & b_z & c_z \end{array}\right|\ . \tag{18}$$

It is clear that if any one of the functions a, b, and c vanishes, i.e. if Z is truely azumuthally symmetric, $\Omega^{(3)}_{\mathrm{CS}}$ vanishes [3].

[3] In particular, if one restricts to the Ansatz

$$Z=\left[\begin{array}{c} a(\rho) \\ c(\rho)\,\mathrm{e}^{in\varphi} \end{array}\right]\equiv\left[\begin{array}{c} \sin\frac{1}{2}f(\rho) \\ \cos\frac{1}{2}f(\rho)\,\mathrm{e}^{in\varphi} \end{array}\right] \tag{19}$$

for the system (11) defined instead on $\mathbb{R}^2$, i.e. replacing the index $i=\alpha,3$ by $\alpha=1,2$, then the solutions are the Belavin-Polyakov (ferromagnet) instantons with vorticity n stabilised by the *first* Chern-Pontryagin charge of the (composite) Abelian connection.

The topological charge can be expressed as

$$Q = 2\pi \int \Omega_{\mathrm{U}(1)}\,\rho\,d\rho\,dz = -2\pi\,n \int \alpha_{[\rho}\,(\cos f)_{z]}\,d\rho\,dz \tag{20}$$

The volume integral (20) can be rewritten using the notation $x_A = (\rho, z)$, $A = 1, 2$ as follows

$$Q = -2\pi\,n \int \varepsilon_{AB}\,\partial_A\alpha\,\partial_B(\cos f)\,d^2x\,, \tag{21}$$

which can be evaluated by applying Stokes' Theorem.

Using a semicircular contour for the line integral resulting from (21) is

$$Q = -2\pi\,n \int_{z=-\infty}^{z=+\infty} \cos f\,\partial_z\alpha\,dz - 2\pi\,n \int_{\theta=0}^{\theta=m\pi} (\cos f\,\partial_\theta\alpha)\,\Big|_{r\to\infty}\,d\theta\,. \tag{22}$$

Now the first integral in (22) should vanish by parity, while the integrand of the second one at infinite $r = \sqrt{\rho^2 + z^2}$ should, since $\lim_{r\to\infty} f(r) = 1$, be equal to

$$\lim_{r\to\infty} \partial_\theta\alpha\,. \tag{23}$$

Thus, requiring the asymptotic value

$$\lim_{r\to\infty} \alpha(r,\theta) = m\,\theta\,, \tag{24}$$

results in

$$Q = -2\pi\,n\,m\,.$$

3.2 Chern–Simons density on $\mathbb{R}^5$

The Chern–Simons density on $\mathbb{R}^5$, denoting the coordinates $x_i = (x_\mu, x_5)$, is

$$\begin{aligned}
\Omega^{(5)}_{\mathrm{U}(1)} &= \varepsilon_{mijkl}\,B_m\,G_{ij}\,G_{kl} \\
&= \varepsilon_{\mu\nu\rho\sigma}\,(B_5\,G_{\mu\nu}\,G_{\rho\sigma} + 4\,B_\mu\,G_{\nu 5}\,G_{\rho\sigma}) \\
&= 2\varepsilon_{\alpha\beta}\varepsilon_{AB}\Big\{B_5\,\left(G_{\alpha\beta}\,G_{AB} - 2G_{\alpha A}\,G_{\beta B}\right) \\
&\quad +2\left[B_\alpha\left(G_{\beta 5}G_{AB} - 2G_{A5}G_{\beta B}\right) + B_A\left(G_{B5}G_{\alpha\beta} - 2G_{\alpha 5}G_{B\beta}\right)\right]\Big\}
\end{aligned} \tag{25}$$

Substituting the bi-azimuthally symmetric Ansatz (16) into (41) yields the simple expression

$$\frac{1}{2}\,\Omega_{\rm CS}^{(5)}=\det\begin{vmatrix} a & b & c & d \\ a_\rho & b_\rho & c_\rho & d_\rho \\ a_\sigma & b_\sigma & c_\sigma & d_\sigma \\ a_z & b_z & c_z & d_z \end{vmatrix}\,. \tag{26}$$

It is clear that if any one of the functions a, b, c and d vanishes, i.e. if Z is truely bi-azumuthally symmetric, $\Omega_{\rm CS}^{(5)}$ vanishes [4].

Substituting the trigonometric parametrisation in (16), namely the parametrisation in which the sigma model constraint is already imposed, (26) reduces to the simple expression

$$\frac{1}{2}\,\Omega_{\rm CS}^{(5)}=4\cdot(2\pi)^2\,n_1\,n_2\int\left[\partial_\rho(\cos f)\,\partial_\sigma g\,\partial_z\alpha+{\rm cycl.}(\rho,\sigma,z)\right]\,d\rho\,d\sigma\,dz\,. \tag{28}$$

Parametrising the coordinates in the notation $(\xi_i=(\rho,\sigma,z))$, $i=1,2,3$, where

$$\xi_i=\begin{pmatrix} r\,\sin\psi\,\sin\theta \\ r\,\sin\psi\,\cos\theta \\ r\,cos\psi \end{pmatrix} \tag{29}$$

with $0\le\psi\le\pi$ and $0\le\theta\le\frac{\pi}{2}$. Then, re-expressing (28) as

$$\begin{aligned}\frac{1}{2}\,\Omega_{\rm CS}^{(5)} &= 4\cdot(2\pi)^2\,n_1\,n_2\int\varepsilon_{ijk}\partial_i(\cos f)\,\partial_j g\,\partial_k\alpha\,d^3\xi \\ &= 4\cdot(2\pi)^2\,n_1\,n_2\int\varepsilon_{ijk}\left((\cos f)\,\partial_j g\,\partial_k\alpha\right)\Big|_{r\to\infty}\hat{\xi}_i\,dS \end{aligned}\tag{30}$$

in an obvious notation where $dS=r^2\,\sin\psi\,d\psi\,d\theta$, and where we have applied Gauss' Theorem.

[4]In contrast to the 3 dimensional case above however, restricting to the Ansatz

$$Z=\begin{bmatrix} a(\rho,\sigma) \\ c(\rho,\sigma)\,e^{in\varphi} \\ d(\rho,\sigma)\,e^{im\chi}\end{bmatrix}\equiv\begin{bmatrix}\sin\frac{1}{2}f(\rho,\sigma) \\ \cos\frac{1}{2}f(\rho,\sigma)\,\sin g(\rho,\sigma)\,e^{in\varphi} \\ \cos\frac{1}{2}f(\rho,\sigma)\,\cos g(\rho,\sigma)\,e^{im\chi}\end{bmatrix} \tag{27}$$

for the system (14) defined instead on $\mathbb{R}^4$, i.e. replacing the index $i=\mu,5$ by $\mu=1,2,3,4$, the resulting solutions are not topologically stable instantons. This is because the *second* Chern-Pontryagin charge exists only for a gauge group containing $SU(2)$.

The result is

$$\frac{1}{2}\,\Omega_{\rm CS}^{(5)}=4\cdot(2\pi)^2\,n_1\,n_2\int_{\psi=0}^{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\cos f\,(\partial_\psi g\,\partial_\theta\alpha-\partial_\psi\alpha\,\partial_\theta g)\Big|_{r\to\infty}d\psi\,d\theta\,. \tag{31}$$

Finally, requiring the boundary values

$$\lim_{r\to\infty} g=\theta\quad,\quad\lim_{r\to\infty}\alpha=m\,\pi\,, \tag{32}$$

(30) yields the following charge

$$\Omega_{\rm CS}^{(5)}=-32\,\pi^3\,n_1\,n_2\,m\,. \tag{33}$$

Note: For $\mathbb{CP}^3$ on $\mathbb{R}^7$, Z must be deformed–tri-azimuthally [5] symmetric, leading to a four dimensional reduced system, and so on. Thus for $\mathbb{CP}^n$ on $\mathbb{R}^{2n+1}$, Z must be deformed–$n-$fold-azimuthally[5] symmetric, leading to a $(n+1)-$dimensional reduced system of PDE's.

The important point here is to realise that the deformed–$n-$fold-azimuthally symmetric Z is encoded with $n+1$ functions, *e.g.* $\alpha, f, g_1, g_2, .., g_{n-2}$ of the n radii of the n planes in $\mathbb{R}^{2n+1}$, plus the $(n+1)-$th compnent. Thus when this symmetry is imposed on the Abelian CS density, one ends up with a total divergence in the residual space. In my opinion, this is a "poor man's" demonstration of the existence of the Hopf charge density.

4 Non-Abelian Grassmannian model on $\mathbb{R}^5$

The sigma model employed is the Grassmannian model described by the complex valued field

$$Z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{34}$$

where z_1 and z_2 are complex 2×2 matrices. The field Z is subject to the constraint

$$Z^\dagger\,Z=\mathbb{I}_{2\times 2}\,. \tag{35}$$

[5] The nomenclature deformed–$n-$fold-azimuthally symmetric here means that first strict $n-$fold-azimuthal symmetry is imposed in the $2n-$dimensional subspace of $\mathbb{R}^{2n+1}$, and then Z is deformed by extending it with the additional phase function α which destroys the $n-$fold-azimuthal symmetry, as in (13) and (16)

Under the action of the local gauge transformation $g \in SU(4)$ on Z we have

$$Z \to Z\,g \qquad \Longrightarrow \qquad D_i Z \to D_i Z\,g \tag{36}$$

where $D_i Z$ is the covariant derivative

$$D_i Z = \partial_i Z - Z\,B_i \tag{37}$$

defined by the *composite* connection

$$B_i = Z^\dagger \partial_i Z\,. \tag{38}$$

Then *composite* (non Abelian) curvature G_{ij} is defined as

$$G_{ij} = \partial_{[i} B_{j]} + [B_i, B_j]\,. \tag{39}$$

Clearly, the gauge group here is $SU(2)$.

It should be remarked at this stage that the Grassmannian sigma model defined by this field on $\mathbb{R}^4$ supports 'instanton' solutions stabilised bt the Pontryagin charge defined in terms the composite curvature G_{ij}, (39). Indeed, if one restricts to the system

$$\mathrm{Tr}\, G_{ij}^2\,, \tag{40}$$

this is scale invariant and the instantons are in addition self–dual

$$G_{ij} = \pm {}^\star G_{ij}\,.$$

Here, we are concerned with systems defined by this field on $\mathbb{R}^5$, and not on $\mathbb{R}^4$. The solutions in this case are not any longer stabilised by the Pontryagin charge, but rather by the Chern-Simons (CS) charge [6]

$$\Omega^{(5)}_{\mathrm{SU(2)}} = \varepsilon_{ijklm} \mathrm{Tr}\, B_m \left[G_{ij} G_{kl} - G_{ij} B_k B_l + \frac{2}{5}\, B_i B_j B_k B_l \right]\,. \tag{41}$$

[6]The analogy with the Abelian case in $\mathbb{R}^3$ is here quite clear. In that case, the CP^1 model on $\mathbb{R}^2$ supports 'instantons' stabilised by the first Pontryagin charge defined in terms of the Abelian composite curvature (9). In particular, when one restricts to the scale invariant system

$$D_i Z^\dagger\, D_i Z\,,$$

the solutions satisfy the first-order self-duality equations

$$D_i Z = \varepsilon_{ij} D_j Z$$

which are the celebrated Belavin–Polyakov vortices modelling the Ferromagnet.

The CP^1 model on $\mathbb{R}^3$ by contrast, does not support instantons stabilised by the Pontryagin, but rather, Hopfions stabilised by the CS charge (17). Of course in that case the system must feature in addition the quartic term G_{ij}^2, for the Derrick scaling requirement to be satisfied. That is the well known Skyrme–Fadde'ev model.

Clearly, the model chosen in this case must differ from (40), which in 5−dimensions would violate the Derrick scaling requirement. To this end one can employ one of the terms

$$\mathrm{Tr} D_i Z^\dagger\, D_i Z \ , \quad \text{and/or} \quad \mathrm{Tr}\, G_{ij}^2\,,$$

and one of

$$\mathrm{Tr}\,(D_{[k} Z^\dagger\, G_{ij]})(G_{[ij}\, D_{k]} Z \ , \quad \text{and/or} \quad \mathrm{Tr}\, G_{ijkl}^2\,.$$

The appropriate symmetry to be imposed in this case is, just as in the CP^2 model on $\mathbb{R}^5$ above, the bi-azimuthal symmetry. The Anzatz resulting fromn this symmetry restricts the functions z_1 and z_2 in (34) to be

$$z_1 \;=\; a(\rho,\sigma,z)\,\mathbb{1} + 2b(\rho,\sigma,z)\,n_\beta m_B\,\Sigma_{\beta B} \tag{42}$$

$$z_2 \;=\; c(\rho,\sigma,z)\,n_\beta\,\tilde{\Sigma}_\beta + d(\rho,\sigma,z)\}m_B\,\tilde{\Sigma}_B \tag{43}$$

where $\Sigma_\mu = (\Sigma_\alpha, \Sigma_A)$ and $\tilde{\S}_\mu = (\tilde{\Sigma}_\alpha, \tilde{\Sigma}_A)$ are the chiral spin matrices of $SO(4)$, so that $\Sigma_{\mu\nu} = (\Sigma_{\alpha\beta}, \Sigma_{\alpha A}, \Sigma_{AB})$ are the Dirac representations $\Sigma_{\mu\nu} = -\frac{1}{4}\tilde{\Sigma}_{[\mu}\tilde{\Sigma}_{\nu]}$.

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