DIRAC COMPLEX, DOLBEAULT COMPLEX, AND SUPERSYMMETRIC QUANTUM MECHANICS

or

ONE MORE PROOF OF THE ATIYAH-SINGER THEOREM

based on [E.Ivanov + A.S., in preparation]

Erevan, August 24, 2010

ATIYAH-SINGER THEOREM

(High school version)

- Consider the motion of a massless electron on the plane in external magnetic field $B(x, y)$.

Dirac operator:

$$\not{D} = \sigma_j(\partial_j - iA_j), \qquad j = 1, 2$$

- $\{\not{D}, \sigma_3\} = 0 \ \rightarrow$

Double degeneracy of all excited level $\equiv$ supersymmetry

.

The index and its integral representation

$$I_{\not{D}} = \mathrm{Tr}\left\{\sigma_3 e^{-\beta \not{D}^2}\right\} = n_L^0 - n_R^0 \; = \\ \Phi = \frac{1}{2\pi}\int B(x,y)\, dxdy \; .$$

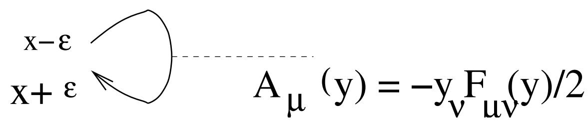

Figure 1: Schwinger-splitted ferm. propagator in external field

Heat kernel proof

- Anomalous divergence

$$\partial_\mu J_\mu \;=\; \frac{1}{4\pi}\epsilon_{\alpha\beta}F_{\alpha\beta}\ .$$

- May be proven by Schwinger splitting

$$J_\mu \to J_\mu(\epsilon) = \bar{\psi}(x+\epsilon)\gamma_\mu\gamma^5\psi(x-\epsilon)$$

Functional integral proof

- Dirac index ≡ Witten index of a SQM system

with

$$H = \not{D}^2, \qquad Q = \not{D}(1+\sigma_3), \qquad \bar{Q} = \not{D}(1-\sigma_3)$$

- can be mapped to

$$H = \frac{1}{2}(P_j + A_j)^2 + \frac{1}{2}B[\psi, \bar{\psi}]$$

then

$$I = \int \prod_\tau \frac{d\bar{\pi}(\tau)d\bar{z}(\tau)d\pi(\tau)dz(\tau)}{(2\pi)^2} d\bar{\psi}(\tau)d\psi^(\tau)$$

$$\exp\left\{\int_0^\beta \left[i\pi\dot{z} + i\bar{\pi}\dot{\bar{z}} + i\dot{\bar{\psi}}\psi - H(\pi, \bar{\pi}, z, \bar{z}; \bar{\psi}, \psi)\right]\right\} ,$$

with periodic boundary conditions.

- For small β, this gives (?) an ordinary integral

$$I = \int \frac{d\pi dz d\bar{\pi} d\bar{z}}{4\pi^2} d\psi d\bar{\psi} \exp\{-\beta H\} \rightarrow \text{magnetic flux},$$

Gen. even-dimensional manifold with Ab. gauge field

$$I = \int e^{\mathcal{F}} \det{}^{-1/2} \left[\frac{\sin \frac{\mathcal{R}}{4\pi}}{\frac{\mathcal{R}}{4\pi}} \right] ,$$

with

$$\mathcal{F} = F_{MN}\, dx^M \wedge dx^N \ , \qquad \mathcal{R}_{MN} = \frac{1}{2} R_{MNPQ}\, dx^P \wedge dx^Q \ .$$

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- Heat kernel proof — Atiyah + Singer 1968,1971

- Functional integral proof — Alvarez-Gaumé, 1983; Friedan + Windey, 1984.

  (based on the standard susy structure $\{\not{D}(1 \pm \sigma_3); \not{D}^2\}$)

- This talk — an alternative proof based on an alternative susy structure for Kähler manifolds.

A SQM MODEL

- Consider the chiral (antichiral) superfields

$$Z^j(t_L,\theta) = z^j(t_L) + \sqrt{2}\theta\psi^j, \quad \bar{Z}^{\bar{i}}(t_R,\bar{\theta}) = \bar{z}^{\bar{j}} - \sqrt{2}\bar{\theta}\bar{\psi}^{\bar{j}} \; .$$

$(t_{L,R} = t \mp i\theta\bar{\theta})$

- Consider the action

$$S = \int dt d^2\theta \, (\mathcal{L} + \mathcal{L}_{WZ}) \, ,$$

$$\mathcal{L} = -\frac{1}{4} g_{i\bar{j}}(Z,\bar{Z}) \, DZ^{\,i} \bar{D}\bar{Z}^{\,\bar{j}} \, , \quad \mathcal{L}_{WZ} = \frac{c_0}{2} \, W(Z,\bar{Z})$$

with

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_t \, , \bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\partial_t$$

- In components:

$$S \equiv \int dt\,(L + L_{WZ}) = \int dt \Big\{ g_{i\bar{j}} \Big[\dot{z}^{\,i} \dot{\bar{z}}^{\,\bar{j}} + \frac{i}{2} \Big(\psi^{\,i} \dot{\bar{\psi}}^{\,\bar{j}} - \dot{\psi}^{\,i} \bar{\psi}^{\,\bar{j}} \Big) \Big]$$

$$- \frac{i}{2} \Big[\Big(2\partial_t g_{i\bar{j}} - \partial_i g_{t\bar{j}} \Big) \dot{z}^{\,i} - \Big(2\partial_{\bar{j}} g_{t\bar{i}} - \partial_{\bar{i}} g_{t\bar{j}} \Big) \dot{\bar{z}}^{\,\bar{i}} \Big] \psi^{\,t} \bar{\psi}^{\,\bar{j}}$$

$$+ (\partial_t \partial_{\bar{l}} g_{i\bar{k}})\, \psi^{\,t} \psi^{\,i} \bar{\psi}^{\,\bar{l}} \bar{\psi}^{\,\bar{k}} + c_0 \Big[\partial_i \partial_{\bar{k}} W\, \psi^{\,i} \bar{\psi}^{\,\bar{k}} - \frac{i}{2} \Big(\partial_i W \dot{z}^{\,i} - \partial_{\bar{i}} W \dot{\bar{z}}^{\,\bar{i}} \Big) \Big] \Big\} .$$

- $g_{i\bar{j}}$ - the metric.

- other terms are expressed via Christoffel symbols, spin connections, torsions, and W.

- For Kähler manifolds,

$$g_{i\bar{k}}(Z,\bar{Z}) = \partial_i \partial_{\bar{k}} K(Z,\bar{Z}) \ ,$$

the torsion terms vanish, and things simplify.

Classical supercharges and hamiltonian

$$Q^K_{cl} = \sqrt{2}\left[\Pi_k - i\bar{\psi}^{\,\bar{a}}\psi^{\,b}\left(\omega_{k,\bar{a}b}\right)\right] e^k_c \psi^{\,c},$$
$$\bar{Q}^K_{cl} = \sqrt{2} e^{\bar{k}}_{\bar{c}} \bar{\psi}^{\,\bar{c}} \left[\bar{\Pi}_{\bar{k}} + i\bar{\psi}^{\,\bar{a}}\psi^{\,d}\left(\bar{\omega}_{\bar{k},d\bar{a}}\right)\right].$$

$$H^K_{cl} = g^{i\bar{k}}\left(\Pi_i - i\omega_{i,\bar{b}a}\,\bar{\psi}^{\,\bar{b}}\psi^{\,a}\right)\left(\bar{\Pi}_{\bar{k}} + i\bar{\omega}_{\bar{k},a\bar{b}}\,\bar{\psi}^{\,\bar{b}}\psi^{\,a}\right)$$
$$-2c_0 e^i_a e^{\bar{k}}_{\bar{b}} \partial_i \partial_{\bar{k}} W \psi^{\,a}\bar{\psi}^{\,\bar{b}} \,,$$

where $\Pi_k = P_k + i(c_0/2)\partial_k W$ and $\omega_{j,\bar{b}a} = e^{\bar{k}}_{\bar{b}} \partial_j e^{\bar{a}}_{\bar{k}}$ are Kähler spin connections.

Quantization

- Ordering ambiguities. Want to keep supersymmetry at quantum level.

- Universal recipe (A.S., 1987):

a) Weyl ordering of classical supercharges gives "flat" supercharges

b) covariant supercharges are obtained by a similarity transformation $Q \to (\det g)^{-1/2} Q (\det g)^{1/2}$.

$$Q^{cov} = \sqrt{2}\psi^c e_c^k \left[\Pi_k - \frac{i}{2}\partial_k(\ln\det\bar{e}) + i\psi^b\bar{\psi}^{\bar{a}}\left(\omega_{k,\bar{a}b}\right)\right]$$

$$\bar{Q}^{cov} = \sqrt{2}\bar{\psi}^{\bar{c}} e_{\bar{c}}^{\bar{k}} \left[\bar{\Pi}_{\bar{k}} - \frac{i}{2}\partial_{\bar{k}}(\ln\det e) + i\bar{\psi}^{\bar{a}}\psi^d\left(\bar{\omega}_{\bar{k},d\bar{a}}\right)\right],$$

COMPLETION TO EXTENDED SUSY KAHLER MODEL

- NO gauge field !
- The Lagrangian can be reduced to

$$\mathcal{L}^K \;=\; -\frac{i}{2}\,\dot{Z}^{\,k}\partial_k\,K$$

(K - Kähler potential)

- Introduce chiral fermionic superfields $\Phi^j, \bar{\Phi}^{\bar{k}}$

and write

$$\tilde{\mathcal{L}}^K = \mathcal{L}^K + \frac{1}{4}\,g_{i\bar{k}}\,\Phi^i\,\bar{\Phi}^{\bar{k}}$$

Bingo !

Geometric interpretation

1. Dolbeault

- Choose

$$W = -(\ln \det g)/(n+1)$$

($\partial_k W$ is called a tautological bundle) and assume $\det \bar{e} = \det e = \sqrt{\det \mathrm{g}}$.

- Let $c_0 = (n+1)/2$. Then

a) Π_k is reduced to a holomorphic derivative and

b) The action of $\hat{Q}$ on the wave functions is isomorphic to the action of the external holomorphic derivatives ∂ on the holom. $(p, 0)$ - forms.

c) $\hat{\bar{Q}}$ maps to $\partial^\dagger$.

- ∂ and $\partial^\dagger$ form the Dolbeault complex.

- $c_0 = -(n+1)/2$.

In this case,

- $\bar{\Pi}_{\bar{k}}$ is reduced to the antiholomorphic derivative
- $\hat{\bar{Q}}$ is mapped to $\bar{\partial}$ and $\hat{Q}$ to $\bar{\partial}^{\dagger}$.
- We obtain the antiholomorphic Dolbeault complex.

- Generic c_0 $\longrightarrow$ twisted Dolbeault and/or anti-Dolbeault complex.

2. Dirac

• Let $c_0 = 0$. Then

$$Q = \sqrt{2}\psi^b e_b^k \left[\partial_k + \frac{1}{2}\omega_{k,\bar{a}d}(\bar{\psi}^{\,\bar{a}}\psi^{\,d} - \psi^{\,d}\bar{\psi}^{\,\bar{a}})\right] .$$

• Map fermion variables to γ - matrices: $\sqrt{2}\psi^a \equiv \gamma^a$, $\sqrt{2}\bar{\psi}^{\bar{a}} \equiv \gamma^{\bar{a}}$. Then

$$Q+\bar{Q} \equiv \not{D} = \gamma^A e_A^M \left(\partial_M + \frac{1}{4}\omega_{M,BC}\gamma^B\gamma^C\right) \equiv \gamma^A \mathcal{D}_A .$$

• Another real supercharge

$$S = i\left[\not{D}^{\mathrm{Hol}} - (\not{D}^{\mathrm{Hol}})^*\right] = \gamma^A I_A^B \mathcal{D}_B ,$$

where $I_A^B, I^2 = -1$, is the matrix of complex structure, $I = \mathrm{diag}(i\sigma_2, \ldots, i\sigma_2)$

• Noticed before by Kirschberg + Lange + Wipf.

- $c_0 \neq 0 \quad \longrightarrow$ Re[Q] is the twisted Dirac operator (with external gauge field)

CONCLUSION:

For Kähler manifolds, the Dirac complex, twisted by a bundle proportional to the tautological bundle $\partial_k \ln \det g$, is equivalent to a twisted holomorphic or antiholomorphic Dolbeault complex.

THE INDEX

- small β limit; functional integral $\to$ ordinary integral,

$$I = \left(\frac{1}{2\pi}\right)^n \int \prod_j dz^j d\bar{z}^{\bar{j}} \det \|g_{i\bar{k}}\| \det \|\mathcal{F}_{a\bar{b}}\| \ ,$$

with $\mathcal{F}_{a\bar{b}} = c_0 e^i_a e^{\bar{k}}_{\bar{b}} \partial_i \partial_{\bar{k}} W$ (generalized magnetic field strength).

- For CP^n, this gives

$$I_{CP^n} \stackrel{?}{=} \frac{(c_0)^n}{n!} \ .$$

- Not integer and strange. Does not take into account curvature.

- The correct result:

$$I_{CP^n} = \begin{pmatrix} c_0 + (n-1)/2 \\ n \end{pmatrix} ,$$

is integer is c_0 is integer (odd n) and half-integer (even n)

Resolution of the paradox : one cannot neglect higher Fourrier modes.
One should instead expand

$$z^j(\tau) = z^{j(0)} + \sum_{m \neq 0} z^{j(m)} e^{2\pi i m \tau / \beta} ,$$

etc. and integrate over $\prod_{jm} dz^{j(m)} \cdots$ in the Gaussian approximation.

• Doing this and going to real notation, we reproduce the known result

$$I = \int e^{\mathcal{F}} \det{}^{-1/2} \left[\frac{\sin \frac{\mathcal{R}}{4\pi}}{\frac{\mathcal{R}}{4\pi}} \right] ,$$

• Origin of $\sin[\cdots]$

$$\prod_{m=1}^{\infty} \frac{(2\pi m)^2}{(2\pi m)^2 + a^2} = \frac{a}{2\sinh(a/2)} .$$