Finite wave propagation speed on metric graphs

R. Schrader
(with V.Kostrykin (Mainz) and J. Potthoff (Mannheim))

Freie Universität Berlin

Erevan, August 2010

Contents

Motivation

.

Experience: Information can be transmitted only with finite speed.
Mathematical framework: Theory of hyperbolic pde's
Important example: The wave equation

$$\Box\psi = 0$$

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta \quad = \text{d'Alembert operator}$$

t = time and $-\Delta$ = Laplace operator on the space under consideration.
Networks in various disguises serve as models for transporting information. So one might ask the question:

Motivation

Can one discuss waves and their propagation in networks?

Is there a mathematical model within which one may ask about finite propagation speed?

Motivation

The answer is *yes*:

The model is given by the following data:

(i) a metric graph $\mathcal{G}$
and
(ii) a Laplace operator $-\Delta$ on $\mathcal{G}$

The metric graph should be viewed as an idealized version of a network

Motivation

The answer is *yes*:

The model is given by the following data:

(i) a metric graph $\mathcal{G}$

and

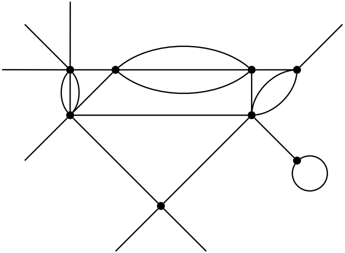
(ii) a Laplace operator $-\Delta$ on $\mathcal{G}$

The metric graph should be viewed as an idealized version of a network

Metric graphs

Definition:

A metric graph $\mathcal{G}$ is a finite collection of half lines and finite intervals of given lengths with an identification of some of their endpoints (=vertices)



A graph with 8 vertices, 7 external and 17 internal edges and one tadpole.

Analysis

So it makes sense to speak of

- Functions on the graph
- Measurable functions
- Lebesgue integration
- Functions which are continuous away from the vertices
- Continuous functions
- Functions which are (infinitely) differentiable away from the vertices
- (Infinitely) Differentiable functions on the graph
- Infinitely differentiable functions away from the vertices and with compact support

Aim and Task

Find and characterize all self-adjoint operators on

$$L^2(\mathcal{G}),$$

the space of all square integrable functions on $\mathcal{G}$,

which formally equal the second derivative.

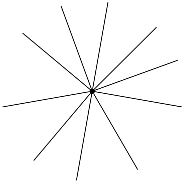
Any of them will be called a

Laplace operator on $\mathcal{G}$

The construction of the Laplace operators will invoke boundary value conditions.

The general one vertex case

is given by an arbitrary number $n = |\mathcal{E}|$ of half lines with one joint vertex



$\mathcal{E}$ = set of external half-lines e, each $\cong [0, \infty)$.

$$\psi \in L^2(\mathcal{G}) \Longleftrightarrow \psi = \{\psi_e(x)\}_{e \in \mathcal{E}} \in \bigoplus_{e \in \mathcal{E}} L^2(\mathbb{R}_+)$$

Construction of the Laplace operators

The Hilbert space is $L^2(\mathcal{G})$
with scalar Product

$$\langle \psi, \varphi \rangle = \sum_{e \in \mathcal{E}} \int_0^\infty \overline{\psi}_e(x) \varphi_e(x) dx.$$

Choose the domain

$$\mathcal{D} = \left\{ \psi \;\middle|\; \psi_e, \psi_e', \psi_e'' \in L^2([0, \infty)) \right\}.$$

Define the operator Δ on $\mathcal{D}$ to be the second derivative
$(\Delta\psi)_e(x) = \psi_e''(x)$.
Also let Δ^0 equal Δ restricted to the domain

$$\mathcal{D}^0 = \left\{ \psi \in \mathcal{D} \;\middle|\; \psi_e(0) = \psi_e'(0) = 0 \right\}.$$

Construction of the Laplace operators

Aim reformulated:
Find all self-adjoint extensions of Δ^0

Δ^0 has defect indices (n, n) (*J. von Neumann*), so the s.a. extensions may be parametrized by the unitary matrix group $\mathrm{U}(n)$.

Alternative discussion in terms of boundary conditions:

Introduce the boundary values for any $\psi \in \mathcal{D}$

$$\underline{\psi} = \{\psi_e(0)\}_{e\in\mathcal{E}} \quad \in \mathbb{C}^n$$

$$\underline{\psi}' = \{\psi'_e(0)\}_{e\in\mathcal{E}} \quad \in \mathbb{C}^n$$

$$[\underline{\psi}] = \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \quad \in \mathbb{C}^{2n}$$

Aim reformulated:
Find all self-adjoint extensions of Δ^0

Δ^0 has defect indices (n, n) (*J. von Neumann*), so the s.a. extensions may be parametrized by the unitary matrix group $\mathrm{U}(n)$.

Alternative discussion in terms of boundary conditions:

Introduce the boundary values for any $\psi \in \mathcal{D}$

$$\underline{\psi} = \{\psi_e(0)\}_{e\in\mathcal{E}} \quad \in \mathbb{C}^n$$

$$\underline{\psi}' = \{\psi_e'(0)\}_{e\in\mathcal{E}} \quad \in \mathbb{C}^n$$

$$[\psi] = \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \qquad \in \mathbb{C}^{2n}$$

Construction of the Laplace operators

We look for a self-adjoint extension Δ of Δ^0 with domain satisfying

$$\mathcal{D}^0 = \mathcal{D}(\Delta^0) \subset \mathcal{D}(\Delta) \subset \mathcal{D}.$$

Set

$$\omega(\psi, \varphi) = \langle \Delta\psi, \varphi \rangle - \langle \psi, \Delta\varphi \rangle,$$

a hermitian symplectic form on $\mathcal{D}$.

Green's Theorem (= integration by parts) relates two hermitian symplectic forms:

$$\omega(\psi, \varphi) = \langle [\psi], J\,[\varphi] \rangle_{\mathbb{C}^{2n}}$$

with

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}.$$

For a self-adjoint extension Δ of Δ_0 this has to vanish for any φ and ψ in the domain $\mathcal{D}(\Delta)$.

Construction of the Laplace operators

We look for a self-adjoint extension Δ of Δ^0 with domain satisfying

$$\mathcal{D}^0 = \mathcal{D}(\Delta^0) \subset \mathcal{D}(\Delta) \subset \mathcal{D}.$$

Set

$$\omega(\psi, \varphi) = \langle \Delta\psi, \varphi \rangle - \langle \psi, \Delta\varphi \rangle,$$

a hermitian symplectic form on $\mathcal{D}$.

Green's Theorem (= integration by parts) relates two hermitian symplectic forms:

$$\omega(\psi, \varphi) = \langle [\psi], J\, [\varphi] \rangle_{\mathbb{C}^{2n}}$$

with

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}.$$

For a self-adjoint extension Δ of Δ_0 this has to vanish for any φ and ψ in the domain $\mathcal{D}(\Delta)$.

Construction of the Laplace operators

Let A and B be two $n \times n$ matrices and define the linear spaces

$$\mathcal{M}(A,B) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^{2n} \;\middle|\; Aa + Bb = 0 \right\} \tag{1}$$

$$\mathcal{D}_{\mathcal{M}(A,B)} = \left\{ \psi \in \mathcal{D} \;\middle|\; [\underline{\psi}] \in \mathcal{M}(A,B) \right\} \supset \mathcal{D}_0 \tag{2}$$

Let $\Delta_{\mathcal{M}(A,B)}$ be the restriction of Δ to $\mathcal{D}_{\mathcal{M}(A,B)}$.

Theorem There is equivalence:

- $\Delta_{\mathcal{M}(A,B)}$ is self-adjoint
- $\mathcal{M}(A,B)$ is a maximal isotropic subspace of $\mathbb{C}^{2n}$
- $AB^\dagger$ is self-adjoint and the $n \times 2n$ matrix (A, B) has maximal rank.

Construction of the Laplace operators

These maximal isotropic subspaces satisfy

(i) $\mathcal{M}(A, B) = \mathcal{M}(A', B')$
iff there is an invertible C with $A' = CA, B' = CB$

(ii) Each maximal isotropic subspace $\mathcal{M} \subset \mathbb{C}^{2n}$ is of this form:
$\mathcal{M} = \mathcal{M}(A, B)$ for some A, B.

(iii) The $n \times n$ (quantum mechanical) scattering matrix at energy $E = \mathrm{k}^2$

$$S_{\mathcal{M}(A,B)}(\mathrm{k}) = -(A + \mathrm{i}\mathrm{k}B)^{-1}(A - \mathrm{i}\mathrm{k}B)$$

is unitary for all real $\mathrm{k} \neq 0$ and meromorphic in the complex k-plane. The poles are located on the imaginary axis and the poles $\mathrm{i}\kappa$ on the positive imaginary axis are via $\varepsilon = -\kappa^2$ in a one to one correspondence with the negative eigenvalues of $-\Delta_{\mathcal{M}(A,B)}$.

(iv) Conversely: For any $\mathrm{k}_0 > 0$ the matrix $S_{\mathcal{M}(A,B)}(\mathrm{k}_0)$ uniquely fixes $\mathcal{M}(A, B)$. Thus the space of all maximal isotropic subspaces of $\mathbb{C}^{2n}$ can be identified with $\mathrm{U}(n)$ (a result obtained independently by V. Arnold).

Construction of the Laplace operators

Extension to a general metric graph $\mathcal{G}$:
$\psi', \psi'' =$ first and second derivatives (away from the vertices).
Operators: (1) $\Delta\psi = \psi''$ with domain

$$\mathcal{D} = \left\{\psi \,\middle|\, \psi, \psi', \psi'' \in L^2(\mathcal{G})\right\}$$

(2) $\Delta^0 =$ restriction of Δ to

$$\mathcal{D}(\Delta^0) = \left\{\psi \in \mathcal{D} \,\middle|\, \psi \text{ and } \psi' \text{ vanish at the vertices}\right\}.$$

Given $\psi \in \mathcal{D}$, the boundary values $\underline{\psi}_v$ and $\underline{\psi}'_v$ (inward normal derivative) of ψ and ψ' at the vertices v again combine to $\underline{\psi}$ and $\underline{\psi}'$ and hence to an element

$$[\underline{\psi}] = \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix}$$

of $C^{2(n+2m)}$, where n denotes the number of external and m the number of internal lines.

Construction of the Laplace operators

Analogously one introduces matrices A_v and B_v as above and maximal isotropic $\mathcal{M}(A_v, B_v)$ spaces in $\mathbb{C}^{2n_v}$ (n_v = number of edges entering v) and correspondingly their direct sum

$$A = \oplus_v A_v,\ B = \oplus_v B_v, M = \oplus_v \mathcal{M}(A_v, B_v)$$

giving rise to a self-adjoint operator $\Delta_{\mathcal{M}(A,B)}$ with domain

$$\mathcal{D}(\Delta_{\mathcal{M}(A,B)}) = \left\{ \psi \in \mathcal{D} \,\middle|\, [\psi] \in \mathcal{M}(A, B) \right\}.$$

The number of negative eigenvalues (with multiplicities) of $-\Delta_{\mathcal{M}(A,B)}$ is smaller or equal to $2(n + 2m)$.
Again: All self-adjoint operators on $L^2(\mathcal{G})$ may be obtained in this way.
They are all finite rank perturbations of each other.

Non-negative $-\Delta$

Let $P_{\mathcal{M}}$ be the orthogonal projection onto $\mathcal{M}$ and $P_{\mathcal{M}_v}$ onto $\mathcal{M}_v$ and set

$$\Omega = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix}$$

$$\Omega_{\mathcal{M}} = P_{\mathcal{M}} \Omega P_{\mathcal{M}} =$$

$$-\begin{pmatrix} -B^\dagger \\ A^\dagger \end{pmatrix} (AA^\dagger + BB^\dagger)^{-1} AB^\dagger (AA^\dagger + BB^\dagger)^{-1} (-B, A)$$

$$= \sum_v P_{\mathcal{M}_v} \Omega P_{\mathcal{M}_v} = \sum_v \Omega_{\mathcal{M}_v},$$

a hermitian $2(n+2m) \times 2(n+2m)$ matrix. Integration by parts gives

$$\langle \varphi, -\Delta_{\mathcal{M}} \psi \rangle_{\mathcal{G}} = \langle \varphi', \psi' \rangle_{\mathcal{G}} + \langle [\varphi], \Omega_{\mathcal{M}} [\psi] \rangle_{\mathbb{C}^{2(n+2m)}}$$

As a consequence $-\Delta_{\mathcal{M}} \geq 0$ if $\Omega_{\mathcal{M}} \geq 0$ if $AB^\dagger \leq 0$.

Wave equation

For any of the above Laplacians $\Delta_{\mathcal{M}}$, by
the spectral representation and operator calculus the
wave operator

$$W_{\mathcal{M}}(t) = \frac{\sin t\sqrt{-\Delta_{\mathcal{M}}}}{\sqrt{-\Delta_{\mathcal{M}}}}$$

is a bounded self-adjoint operator for all real t as is
$\partial_t W_{\mathcal{M}}(t) = \cos t\sqrt{-\Delta_{\mathcal{M}}}$.

For any Cauchy data $\psi, \dot{\psi} \in L^2(\mathcal{G})$

$$\psi(t) = \partial_t W_{\mathcal{M}}(t)\psi + W_{\mathcal{M}}(t)\dot{\psi}$$

is a solution of the wave equation

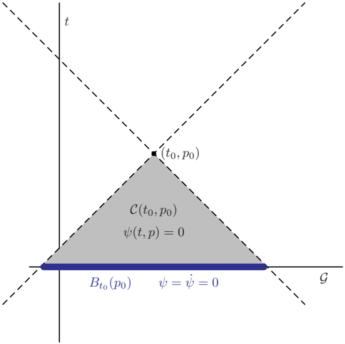
$$\Box_{\mathcal{M}}\varphi(t) = (\partial_t^2 - \Delta_{\mathcal{M}})\psi(t) = 0.$$

satisfying the initial conditions $\psi(t=0) = \psi, \partial_t\psi(t=0) = \dot{\psi}$.

Finite Propagation Speed

Fix a point p_0 and let $B_t(p_0)$ denote the set of points in $\mathcal{G}$ with distance less or equal to t from p_0. Fix $t_0 > 0$ and define the space-time cone

$$\begin{aligned}\mathcal{C}(t_0, p_0) &= \{(t, q) \mid dist(p_0, q) \leq t_0 - t, q \in \mathcal{G}, 0 \leq t \leq t_0\} \subset \mathbb{R}_+ \times \mathcal{G} \\ &= \bigcup_{0 \leq t \leq t_0} B_{t_0 - t}(p_0)\end{aligned}$$



Finite Propagation Speed

Theorem (finite propagation speed)

If both ψ and $\dot{\psi}$ vanish on $B_{t_0}(p_0)$, then $\psi(t)(q)$ vanishes for all $(t, q) \in \mathcal{C}(t_0, p_0)$.

Proof of Finite Propagation Speed

The proof follows the standard line but uses an additional new and crucial ingredient:
For *any* solution $\psi(t)$ introduce the local energy functional

$$0 \leq e(t) = \int_{q \in B_{t_0 - t}(p_0)} \left(|\psi(t)'(q)|^2 + |\partial_t \psi(t)(q)|^2 \right) \, dq$$
$$+ \sum_{v \in B_{t_0 - t}(p_0)} \langle [\psi(t)], \Omega_{\mathcal{M}_v} [\psi(t)] \rangle_{\mathbb{C}^{2(n+2m)}}.$$

The first term is familiar from (text book) proofs of finite propagation speed on smooth manifolds. The second boundary contribution is the new ingredient.

This function is piecewise differentiable and monotonically decreasing in t. To establish this one first has to establish that the boundary values $[\psi(t)]$ are continuously differentiable in t. This follows by using the assumptions $\psi \in \mathcal{D}((-\Delta)^2), \dot{\psi} \in \mathcal{D}((-\Delta)^{3/2})$ and Sobolev estimates. For t where $e(t)$ is differentiable, on computes $\partial_t e(t) \leq 0$. At the other points one uses

$$\sum_{v \in B_{t_0 - t}(p)} \Omega_{\mathcal{M}_v} \leq \sum_{v \in B_{t_0 - t'}(p)} \Omega_{\mathcal{M}_v} \quad \text{for} \quad t \geq t'.$$

But by assumption on the Cauchy data $e(t = 0) = 0$. Since $e(t) \geq 0$ is monotonically decreasing, this implies $e(t) = 0$ for all $0 \leq t \leq t_0$. The definition of $e(t)$ now implies $\psi(t)(p) = 0$ in the cone $\mathcal{C}(t_0, p_0)$ thus concluding the proof.

(i) Can the condition $\Omega_{\mathcal{M}} \geq 0$ be dropped?

For a single vertex graph, finite propagation speed can be proved for all Laplacians $\Delta_{\mathcal{M}}$.

Consider a metric graph $\mathcal{G}$, which is isometrically embedded in $\mathbb{R}^3$, say, and let $\mathcal{T}(\mathcal{G})$ be a tubular neighborhood of $\mathcal{G}$ with smooth boundary. Thus $\mathcal{G}$ is a singular deformation retract of $\mathcal{T}(\mathcal{G})$.

(ii) Can one relate wave propagation on $\mathcal{T}(\mathcal{G})$ to one on $\mathcal{G}$?