Between Continuous and Discrete in Integrable Systems

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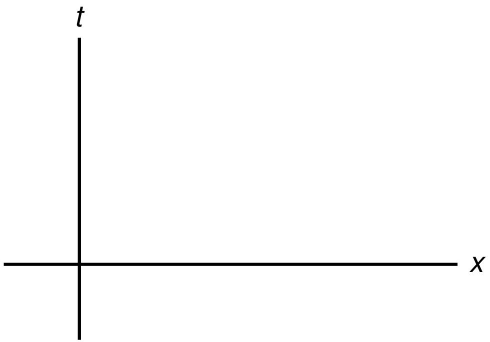
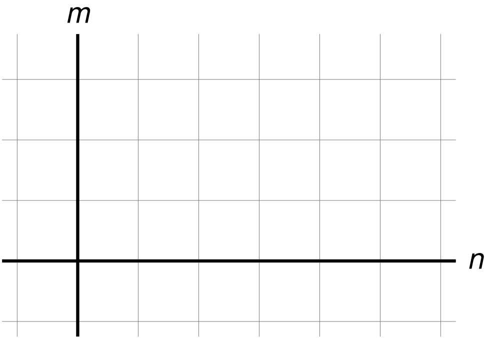
Discrete Integrable Systems - Introduction

KdV: field $u(x, t)$



$$u_t = \frac{1}{4} u_{xxx} + 3uu_x$$

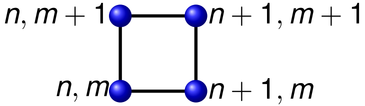
Hirota Discrete KdV (HDKdV): field $u_{n,m}$ defined on rectangular grid



$$\frac{1}{u_{n+1,m+1}} - \frac{1}{u_{n,m}} = C\left(u_{n+1,m} - u_{n,m+1}\right)$$

A *quad-graph equation*: relates values of the field at vertices of a fundamental rectangular lattice domain.

$$\frac{1}{u_{n+1,m+1}} - \frac{1}{u_{n,m}} = C\left(u_{n+1,m} - u_{n,m+1}\right)$$



Write $\frac{1}{u_{11}} - \frac{1}{u_{00}} = C\left(u_{10} - u_{01}\right)$ for short

KdV and HDKdV share common algebraic structure.
Can obtain KdV as a continuum limit of HDKdV:

$$
\begin{aligned}
u_{n,m} &\rightarrow 1 + h^2 u\left(x - \frac{h}{2}, t - \frac{h^3}{2}\right) \\
u_{n+1,m} &\rightarrow 1 + h^2 u\left(x + \frac{h}{2}, t - \frac{h^3}{2}\right) \\
u_{n,m+1} &\rightarrow 1 + h^2 u\left(x - \frac{h}{2}, t + \frac{h^3}{2}\right) \\
u_{n+1,m+1} &\rightarrow 1 + h^2 u\left(x + \frac{h}{2}, t + \frac{h^3}{2}\right)
\end{aligned}
$$

$$
u_t = \frac{C+1}{h^2(C-1)} u_x + \frac{C+1}{24(C-1)} u_{xxx} + \frac{2}{C-1} u u_x + O(h^2)
$$

HDKdV in fact an *integrable quad-graph equation*.

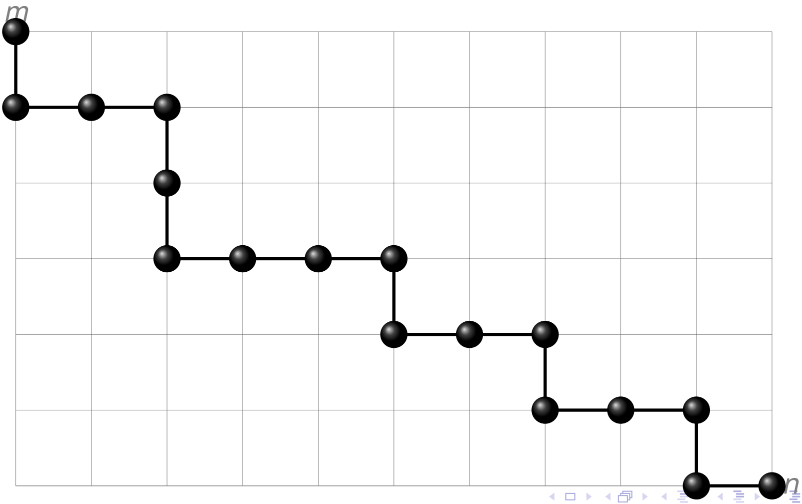
First appeared in 1970s — but active area of research in last decade.

Various criteria for integrability:

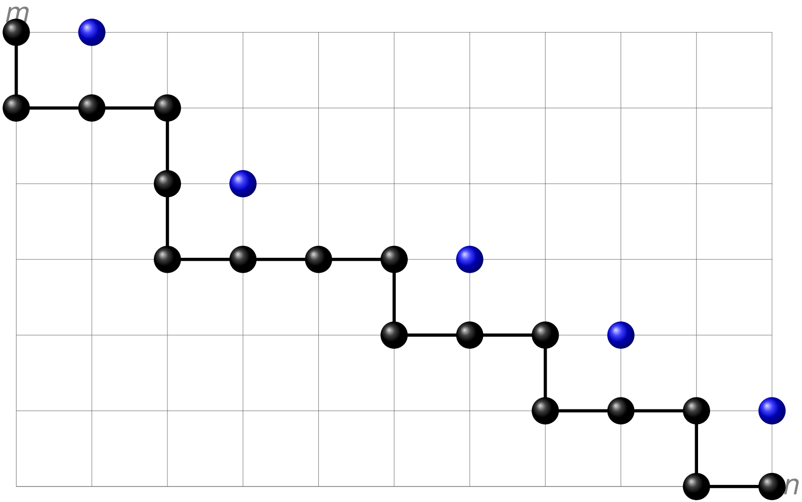
- Lax pairs
- Consitency on a cube
- Singularity confinement (discrete Painlevé propert y)

Rich theory but few known applications.

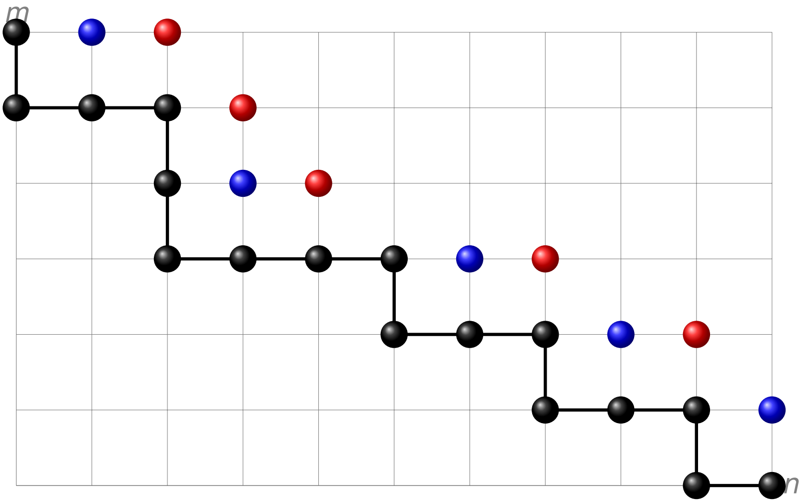
Most authors study quad-graph equations with initial values on a staircase



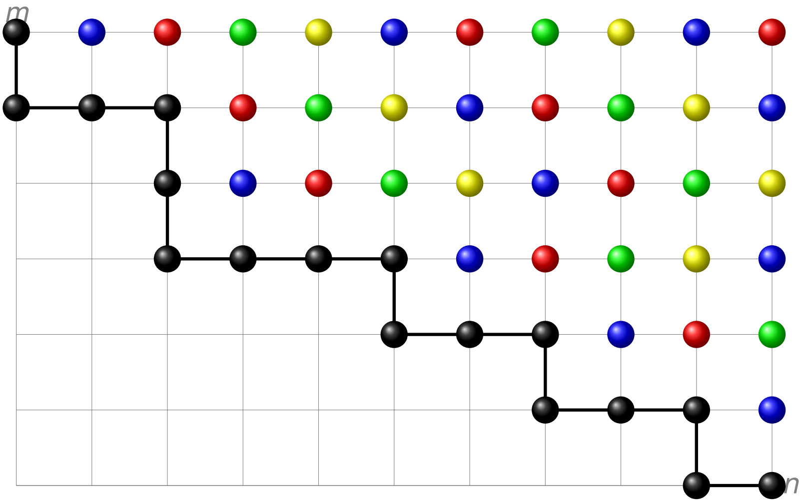
Step 1:



Step 2:



Steps 3,4,5,6,7,8,9,10:



Other authors interested in evolution in m

- Either $\{u_{n,0}\}_{n\in\mathbf{Z}}$ given with suitable conditions as $|n| \to \infty$, want to find $\{u_{n,1}\}_{n\in\mathbf{Z}}$, $\{u_{n,2}\}_{n\in\mathbf{Z}}$, etc. with same boundary conditions.
- Or $\{u_{n,0}\}_{0\leq n\leq N}$ given with periodic boundary conditions $u_{n,0} = u_{n,N}$ and want to propagate this.

Much harder! But analogous to continuum KdV.

General types of question to ask for integrable quad-graph equations:

- Verify that properties of continuous integrable systems have discrete analogs.
- Find novel features that do not have continuum analogs, and investigate origins and implications of these features.
- Find useful discrete versions of familiar continuous integrable systems for application, for example, in numerical simulations.

This talk

- Proof that lattice KdV has an infinite number of independent conserved quantities (with Sasha Rasin, 2009, JPA) - a discrete version of the Gardner transform for continuous KdV.
- A comment on the family of equations related by Miura maps to HDKdV/lattice KdV — larger than the analogous family for continuum KdV because of a novel symmetry. (Work in progress.)
- “Full lattice KdV”, my own version of “discrete KdV" (Nonlinearity 2003)

Conservation laws of lattice KdV

Lattice KdV:

$$(u_{n+1,m+1} - u_{n,m})(u_{n+1,m} - u_{n,m+1}) = \beta - \alpha$$

or

$$(u_{11} - u_{00})(u_{10} - u_{01}) = \beta - \alpha$$

Exercise: If u satisfies lattice KdV, $u^H_{n,m} = u_{n,m} - u_{n-1,m-1}$ satisfies HDKdV with $C = \frac{1}{\beta-\alpha}$.

Conservation laws of KdV $u_t = \frac{1}{4}u_{xxx} + 3uu_x$

A conservation law is an expression

$$\partial_t G + \partial_x F = 0$$

that holds as a result of the equation. F, G functions of x, t, u and a finite number of partial derivatives of u.

The first 3:

$$\begin{aligned}
\partial_t (u) + \partial_x \left(-\frac{1}{4}u_{xx} - \frac{3}{2}u^2 \right) &= 0 , \\
\partial_t \left(u^2 \right) + \partial_x \left(-\frac{1}{2}uu_{xx} + \frac{1}{4}u_x^2 - 2u^3 \right) &= 0 , \\
\partial_t \left(4u^3 - u_x^2 \right) + \partial_x \left(-9u^4 + \frac{u_x u_{xxx}}{2} - \frac{u_{xx}^2}{4} - 3u^2 u_{xx} + 6uu_x^2 \right) &= 0 .
\end{aligned}$$

A conservation law is trivial if it can be written in the form

$$\begin{aligned} F &= F_0 - \partial_t f \\ G &= G_0 + \partial_x f \end{aligned}$$

where F_0, G_0 both vanish as a consequence of the equation and f is an arbitrary function of x, t, u and a finite number of partial derivatives of u.

For KdV and for all equations of the form $u_t = \ldots$ it is easy to recognize trivial conservation laws. Use the equation to eliminate all t derivatives in G and see if the result is a total derivative with respect to x.

Bäcklund transformation for KdV. If u is a solution so is $u + v_x$ where

$$\begin{aligned} v_x &= \theta - 2u - v^2 \\ v_t &= -\frac{1}{2}u_{xx} + (\theta + u)\left(\theta - 2u - v^2\right) + u_x v \end{aligned}$$

Check that

$$\partial_t v + \partial_x \left(\frac{1}{2}u_x - (u+\theta)v\right) = 0 \ .$$

So v is the G-component of a conservation law. And easy to find formal expansion of v in powers of θ:

$$\begin{aligned} v &= \theta^{1/2} - \frac{u}{\theta^{1/2}} + \frac{u_x}{2\theta} - \frac{u_{xx} + 2u^2}{4\theta^{3/2}} + \frac{u_{xxx} + 8uu_x}{8\theta^2} \\ &\quad - \frac{u_{xxxx} + 8u^3 + 10u_x^2 + 12uu_{xx}}{16\theta^{5/2}} + O\left(\theta^{-3}\right) \ . \end{aligned}$$

- All the integer powers of θ give trivial conservation laws. The half-integer powers give nontrivial conservation laws.
- Straightforward to compute the coefficient of u^n in the coefficient of $\theta^{\frac{1}{2}-n}$ and in particular verify it is not zero: so there are an infinite number of nontrivial conservation laws.
- Can this be done for discrete?

Conservation laws for quad-graph equations. An expression of the form

$$(\mathrm{Up} - I)G + (\mathrm{Right} - I)F = 0$$

that holds as a result of the equation. F, G functions of n, m and values of u at a finite number of lattice sites.

Trivial if

$$\begin{aligned} F &= F_0 - (\mathrm{Up} - I)f \\ G &= G_0 + (\mathrm{Right} - I)f \end{aligned}$$

where F_0, G_0 both vanish as a consequence of the equation and f an arbitrary function of n, m and values of u at a finite number of lattice sites.

We say a conservation law is “on the (horizontal) line” if G only depends on the values of u for fixed m.

In general there is no easy way to recognize trivial conservation laws in the discrete case. But a conservation law on the line is trivial if and only if $G = (\mathrm{Right} - I)f$ for some f on the line.

Showing lattice KdV has an infinite number of conservation laws on the line:
Start from the Bäcklund transformation $u \to \tilde{u}$ given by

$$\begin{aligned}(\tilde{u}_{00} - u_{01})(u_{00} - \tilde{u}_{01}) &= \theta - \beta\ , \\ (\tilde{u}_{00} - u_{10})(u_{00} - \tilde{u}_{10}) &= \theta - \alpha\ .\end{aligned}$$

Two trivial solutions:

$$\begin{aligned}\theta &= \alpha & \qquad \tilde{u}_{00} &= u_{10} \\ \theta &= \beta & \qquad \tilde{u}_{00} &= u_{01}\end{aligned}$$

Expand around the first: $\theta = \alpha + \epsilon$,

$$\tilde{u}_{00} = u_{10} + \sum_{i=1}^{\infty} v_{00}^{(i)} \epsilon^i\ .$$

Find

$$v_{00}^{(1)} = \frac{1}{u_{00} - u_{20}}$$

and for $i > 1$

$$v_{00}^{(i)} = \frac{1}{u_{00} - u_{20}} \sum_{j=1}^{i-1} v_{00}^{(j)} v_{10}^{(i-j)}$$

All these formulas are "on the line".

Can we write a "universal" conservation law in terms of $\tilde{u}$ as in continuum case? Turns out that we can:

$$F = -\ln\left(\tilde{u}_{00} - u_{01}\right)\ , \quad G = \ln\left(\tilde{u}_{00} - u_{10}\right)$$

Expand these in terms of ϵ:

$$F = \sum_{i=0}^{\infty} F_i \epsilon^i\ , \quad G = \ln\epsilon + \sum_{i=0}^{\infty} G_i \epsilon^i$$

and find ....

$$\begin{cases} F_0 = \ln B \\ G_0 = \ln A_0 \end{cases},$$

$$\begin{cases} F_1 = -BA_0 \\ G_1 = A_0A_1 \end{cases},$$

$$\begin{cases} F_2 = -A_0^2A_1B + \frac{1}{2}A_0^2B^2 \\ G_2 = A_0A_1^2A_2 + \frac{1}{2}A_0^2A_1^2 \end{cases},$$

$$\begin{cases} F_3 = -A_0^2A_1^2A_2B - A_0^3A_1^2B + A_0^3A_1B^2 - \frac{1}{3}A_0^3B^3 \\ G_3 = A_0A_1^2A_2^2A_3 + A_0^2A_1^3A_2 + A_0A_1^3A_2^2 + \frac{1}{3}A_0^3A_1^3 \end{cases} \quad \text{etc.}$$

where

$$A_i = \mathrm{Right}^i\left(\frac{1}{u_{00} - u_{20}}\right), \quad i = 0, 1, 2, \dots, \qquad B = \frac{1}{u_{10} - u_{01}},$$

Independence easy using a homogeneity argument.
What about nontriviality? Use a simple continuum limit: In

$$(u_{11} - u_{00})(u_{10} - u_{01}) = \beta - \alpha$$

Set

$$\begin{aligned} u_{00} &= u(x,t) & u_{10} &= u(x+h,t) \\ u_{01} &= u(x,t+h) & u_{11} &= u(x+h,t+h) \\ \alpha &= \alpha(h) & \beta &= \beta(h) \end{aligned}$$

and do $h \to 0$ limit to get eikonal equation

$$u_x^2 - u_t^2 = C \ , \qquad C = \lim_{h \to 0} \frac{\beta(h) - \alpha(h)}{h^2}$$

Easy to check nontriviality of conservation laws in limit for this equation.

The lattice KdV family

The KdV family: Equations related to KdV:

$$\begin{array}{ll} \text{KdV} & u_t = \frac{1}{4}u_{xxx} + 3uu_x \\ \text{PKdV} & b_t = \frac{1}{4}b_{xxx} + \frac{3}{2}b_x^2 \\ \text{Hirota} & 4(\tau_{xt}\tau - \tau_x\tau_t) = \tau\tau_{xxxx} - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 \end{array}$$

$$u = b_x = (\ln\tau)_{xx}$$
$$b = \frac{\tau_x}{\tau}$$

$$\begin{array}{ll} \text{KdV} & u_t = \frac{1}{4}u_{xxx} + 3uu_x \\ \text{MKdV} & v_t = \frac{1}{4}v_{xxx} - \frac{3}{8}v^2 v_x \\ \text{PMKdV} & h_t = \frac{1}{4}h_{xxx} - \frac{1}{8}h_x^3 \\ \text{UrKdV} & q_t = \frac{1}{4}\left(q_{xxx} - \frac{3}{2}\frac{q_{xx}^2}{q_x}\right) \end{array}$$

$$u = \frac{1}{4}v_x - \frac{1}{8}v^2 = \frac{1}{4}\left(\frac{q_{xxx}}{q_x} - \frac{3}{2}\frac{q_{xx}^2}{q_x^2}\right)$$
$$v = h_x = \frac{q_{xx}}{q_x}$$
$$h = \ln q_x$$

The latter “tower” arises from the fact that KdV is the consistency condition for the linear system

$$
\begin{aligned}
\psi_{xx} &= -2u\psi \\
\psi_t &= -\frac{1}{2}u_x\psi + u\psi_x
\end{aligned}
$$

This has two linearly independent solutions ψ_1, ψ_2 only determined up to the obvious action of $GL(2)$. Eliminating u, both satisfy

$$\psi_t = \frac{1}{4}\psi_{xxx} - \frac{3}{4}\frac{\psi_x\psi_{xx}}{\psi} \ .$$

Easy to check $q = \frac{\psi_1}{\psi_2}$ satisfies UrKdV — only determined up to the obvious $PSL(2)$ action. Passing from q to h to v to u is equivalent to modding out by the $PSL(2)$ action, one dimension at a time.

HDKdV

$$\frac{1}{u_{11}} - \frac{1}{u_{00}} = C\,(u_{10} - u_{01})$$

Numerous other equations in the literature related to this by Miura maps (or Bäcklund transformations) — we have seen lattice KdV (analog of PKdV).

HKdV is the consistency condition of

$$\begin{aligned} \psi_{20} &= \psi_{00} + \left(\frac{1}{u_{10}} + Cu_{00}\right)\psi_{10} \\ \psi_{01} &= \frac{\psi_{00}}{u_{00}} - \psi_{10} \end{aligned}$$

There are two linearly dependent solutions of this system, $\psi^{(1)}$,$\psi^{(2)}$, only determined up to the obvious $GL(2)$ action.

- Both $\psi^{(1)}$,$\psi^{(2)}$ satisfy

$$(\psi_{00}+\psi_{11})(\psi_{01}+\psi_{10}) = -C\psi_{00}\psi_{10}$$

— analog of PMKdV. A solution of this gives a solution of HDKdV via

$$u_{00} = \frac{\psi_{00}}{\psi_{01}+\psi_{10}}$$

- $v_{00} = \frac{\psi_{10}}{\psi_{00}}$ satisfies a complicated quad-graph equation, analog of MKdV. There is a $\psi \to v \to u$ tower.
- $q_{00} = \frac{\psi^{(1)}_{00}}{\psi^{(2)}_{00}}$, defined up to the obvious $PSL(2)$ action, satisfies the *cross-ratio equation*

$$\frac{(q_{00}-q_{10})(q_{01}-q_{11})}{(q_{00}-q_{01})(q_{10}-q_{11})} = C+1$$

But this does *not* sit at the bottom of the tower.

Why not?
HDKdV

$$\frac{1}{u_{11}} - \frac{1}{u_{00}} = C\,(u_{10} - u_{01})$$

has an extra (continuous) symmetry

$$u_{nm} \to \begin{cases} \lambda u_{nm} & n+m \text{ even} \\ \frac{1}{\lambda} u_{nm} & n+m \text{ odd} \end{cases}$$

Under this symmetry ψ transforms via

$$\psi_{nm} \to \begin{cases} \sqrt{\lambda}\psi_{nm} & n+m \text{ even} \\ \frac{1}{\sqrt{\lambda}} \psi_{nm} & n+m \text{ odd} \end{cases}$$

But $q = \frac{\psi^{(1)}}{\psi^{(2)}}$ is an invariant! So it is not possible to build ψ or u out of q.

It is possible to mod out the extra symmetry. Define

$$U_{00} = \frac{1 - Cu_{00}u_{10}}{1 + Cu_{00}u_{10}}$$

A simple calculation shows that U satisfies the Levi-Yamilov I equation

$$(U_{10} + 1)(U_{00} - 1) = (U_{01} + 1)(U_{11} - 1)$$

Similarly, modding out PMKdV by the new symmetry yields LY II.
Further, there is a Miura map from the cross ratio equation to LY II and to LY I.

$$
\begin{array}{ccc}
\text{HDKdV} & \longrightarrow & \text{LY I} \\
\uparrow & & \\
\text{PMKdV} & & \uparrow \\
\uparrow & & \\
\text{PDMKdV} & \longrightarrow & \text{LY II} \\
 & & \uparrow \\
 & & \text{cross ratio eq}
\end{array}
$$

Full lattice KdV

(J.S. Nonlinearity 2003)
The linear system for which KdV is the consistency condition (Lax pair) arises via "dressing" of the simple linear system

$$\partial_x U = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U$$

$$\partial_t U = \lambda \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U$$

The Lax pair for Lattice KdV arises via dressing of the discrete version

$$U_{n+1,m} = \left(I + h \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right) U_{n,m}$$

$$U_{n,m+1} = \left(I + k \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right) U_{n,m}$$

A more “useful” discrete, integrable version of KdV should arise via the dressing procedure from the system

$$U_{n+1,m} = \left(I + h \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right) U_{n,m}$$

$$U_{n,m+1} = \left(I + k\lambda \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right) U_{n,m}$$

It works, but the system is complicated: Three fields b, β, Δ:

$$\beta_{n+1,m} + \beta_{n,m} = \frac{b_{n+1,m} + b_{n+1,m+1} - b_{n,m} - b_{n,m+1}}{h} + (b_{n+1,m+1} - b_{nm})(b_{n,m+1} - b_{n+1,m})$$

$$\Delta_{n+1,m} + \Delta_{n,m} = \left(\frac{\beta_{n+1,m} - \beta_{n,m}}{h}\right)\left(-1 + \frac{h}{2}(b_{n+1,m+1} + b_{n+1,m} - b_{n,m+1} - b_{n,m})\right)$$

$$\left(\frac{\beta_{n+1,m} - \beta_{n,m}}{h}\right)\left(\frac{b_{n,m+1} + b_{n+1,m+1} - b_{n,m} - b_{n+1,m}}{k}\right) = \frac{\sqrt{1 + k^2(\Delta^2_{n+1,m} + \beta^3_{n+1,m})} - \sqrt{1 + k^2(\Delta^2_{n,m} + \beta^3_{n,m})}}{\frac{1}{2}hk^2}$$

Do normal continuum limit

$$b_{n,m} = b(x,t) \qquad b_{n+1,m} = b(x+h,t)$$
$$b_{n,m+1} = b(x,t+k) \qquad b_{n+1,m+1} = b(x+h,t+k)$$

Take $h,k \to 0$ limit to get

$$2\beta = 2b_x \ , \qquad 2\Delta = -\beta_x \ , \qquad 2\beta_x b_t = (\Delta^2 + \beta^3)_x \ .$$

Eliminate β and Δ to find PKdV

$$b_t = \frac{1}{4} b_{xxx} + \frac{3}{2} b_x^2$$

Could be implemented as a numerical method. But more significant is that we now have the tools to find the conservation laws of this discrete system and compare them against the continuum conservation laws. To date there is no known numerical method for KdV that has bounded errors in the first 3 conservation laws, this may provide it.

Thank you

Thank you for listening!