Deformation Quantization of Instantons

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0. Introduction

Notations : Comm. relation, Moyal product

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \;\; \mu, \nu = 1, \ldots, 2n \;,$$

$(\theta^{\mu\nu})$: real, x-indep., skew-sym., NC parameters.

$$f(x) \star g(x) = f(x)g(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f(x) \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right)^n g(x) \;.$$

Introduce $\hbar$ and a fixed constant $\theta_0^{\mu\nu} < \infty$ with

$$\theta^{\mu\nu} = \hbar \theta_0^{\mu\nu}$$

We define the commutative limit by letting $\hbar \to 0$.

The curvature two form F :

$$F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge \star dx^\nu = dA + A \wedge \star A$$

where $A \wedge \star A := \frac{1}{2}(A_\mu \star A_\nu) dx^\mu \wedge dx^\nu$.
Instanton is defined by

$$F^+ = \frac{1}{2}(1 + *)F = 0 \ ,$$

$*$: Hodge star.

NC instantons in $\mathbb{R}^4$ are discovered by the ADHM method (Nekrasov-Schwarz).
Many works was done. (Lechtenfeld, Szabo, . . .)

[Known facts of ADHM instantons]:
ADHM data $\Longrightarrow$ Instanton (including U(1))
NC ADHM instanton $\sharp = k$
It does not depend on the NC parameter
(A.S. -Ishikawa -Kuroki, A.S.,Furuuchi,Tian)
(These are same as comm. instanton)

[Can we expect that ?]:

1. Instanton $\sharp$ are inv. under NC deform. in $\mathbb{R}^4$?
2. Top. charges in Y-M are preserved in $\mathbb{R}^n$??
   (Vortex, Monopole and so on.)
3. "ADHM data $\Longleftrightarrow$ NC Instanton " is 1 to 1?
4. Can we make U(1) Instanton as Deformation?

(1) NC Deformation of Instantons

Formally we expand A as $A_\mu = \sum_{l=0}^{\infty} A_\mu^{(l)} \hbar^l$.

Using $P := \frac{1+*}{2}$ and covariant derivatives associated to $A_\mu^{(0)}$ by $D^{(0)} f := d\, f + A^{(0)} \wedge f$

l-th order Instanton Eq.

$$P(D^{(0)} A^{(l)} + C^{(l)}) = 0.$$

where

$$C^{(l)}_{\rho\tau} := \sum_{(p;\ m,n)\in I(l)} \hbar^{p+m+n}\frac{1}{p\ !}\left(A^{(m)}_{[\rho}(\overleftrightarrow{\Delta})^{p}A^{(n)}_{\tau]}\right)$$

$$\overleftrightarrow{\Delta} \equiv \frac{i}{2}\overleftarrow{\partial}_{\mu}\theta^{\mu\nu}_{0}\overrightarrow{\partial}_{\nu}.$$

$$I(l) \equiv \{(p;\ m,n) \in \mathbb{Z}^{3} | p+m+n = l, m \neq l, n \neq l\}.$$

Note that :

- $C^{(l)}_{\rho\tau}$ is consisted of $A^{(k)}$ $(k < l)$. i.e. given fun.
  We determine $A^{(l)}$ recursively.
- 0-th order is the comm. instanton Eq.

Asymptotic behavior of comm. instanton $A_\mu^{(0)}$

$$A^{(0)} = gdg^{-1} + O(|x|^{-2}),\ gdg^{-1} = O(|x|^{-1}),$$

where $g \in G$ and G is a gauge group.
Fix $A^{(0)}$ and impose a condition for $A^{(l)}(l \geq 1)$ as

$$A - A^{(0)} = D^*_{A^{(0)}} B\ ,\ B \in \Omega^2_+,$$

where $D^*_{A^{(0)}}$ is defined by

$$(D^*_{A^{(0)}})_\rho^{\mu\nu} B_{\mu\nu} \ = \ \delta_\rho^\nu D^{(0)\mu} B_{\mu\nu} - \delta_\rho^\mu D^{(0)\nu} B_{\mu\nu}.$$

This is chosen to deform the Eq. into elliptic DE.

We expand B in $\hbar$ as $B = \sum B^{(k)}\hbar^k$.
Using the fact that the $A^{(0)}$ is anti-selfdual,

$$\boxed{2D^2_{(0)}B^{(l)\mu\nu} + P^{\mu\nu,\rho\tau}C^{(l)}_{\rho\tau} = 0,} \quad : \quad \text{Main Eq.}$$

where

$$D^2_{(0)} \equiv D^{\rho}_{A^{(0)}} D_{A^{(0)}\rho} \; .$$

The Green's fun.: $D^2_{(0)}G_0(x,y) = \delta(x-y)$,
$G_0(x,y)$ was constructed (Corrigan et.al). Then,

$$B^{(l)\mu\nu} = -\frac{1}{2}\int_{\mathbb{R}^4} G_0(x,y)P^{\mu\nu,\rho\tau}C^{(l)}_{\rho\tau}(y)d^4y$$

and the NC instanton $A = \sum A^{(l)} \hbar^l$ is given by

$$A^{(l)} = D^*_{A^{(0)}} B^{(l)}.$$

Using this, we can prove

$$\boxed{|A^{(l)}| < O(|x|^{-3+\epsilon}), \quad \forall \epsilon > 0}$$

By using this fact, we can prove the following Theorems.

(2) Instanton $\sharp$ is indep. of $\hbar$

$$\frac{1}{8\pi^2}\int trF\wedge\star F=\frac{1}{8\pi^2}\int trF^{(0)}\wedge F^{(0)},$$

Summarizing the above discussions,

Theorem 1. *Let $A_\mu^{(0)}$ be a comm. instanton in $\mathbb{R}^4$. There exists a formal NC instanton $A_\mu=\sum_{l=0}^{\infty}A_\mu^{(l)}\hbar^l$ such that the instanton number is independent of the NC parameter $\hbar$.*

(3) Index of Dirac Op.

The Weitzenbock formula shows that

$$\bar{\mathcal{D}}_A \star \mathcal{D}_A = \Delta_A + \sigma^+ F^+ \ ,$$
$$\mathcal{D}_A \star \bar{\mathcal{D}}_A = \Delta_A + \sigma^- F^- \ ,$$

where $\sigma^+ F^+ = 2\bar{\sigma}^{\mu\nu} F^+_{\mu\nu}$, $\sigma^- F^- = 2\sigma^{\mu\nu} F^-_{\mu\nu}$ and $\Delta_A = D^\mu \star D_\mu$.

Lemma 2. $Ker\mathcal{D}_A\star = 0$ *for* L^2 *function. i.e.* $\psi = 0$ *if* $\mathcal{D}_A \star \psi = 0$ $(\psi^{(l)} \in L^2)$.

Next, we investigate the zero modes of $\bar{\mathcal{D}}_A$.
$\hbar$ expansion of $\bar{\psi} \in \Gamma(S^- \otimes E)[[\hbar]]$

$$\bar{\psi} = \sum_{n=0}^{\infty} \hbar^n \bar{\psi}^{(n)}$$

The 0-th order eq. of $\bar{\mathcal{D}}_A \star \bar{\psi} = 0$ is $\bar{\mathcal{D}}_A^{(0)} \bar{\psi}^{(0)} = 0$.
There are k zero-mode for $A^{(0)}$: $\bar{\psi}_i^{(0)} (i = 1, \dots, k)$
Denote by $\bar{\psi}_i = \sum_{n=0}^{\infty} \hbar^n \bar{\psi}_i^{(n)}$ the zero modes.

The n-th order equation of $\bar{\mathcal{D}}_A \star \bar{\psi} = 0$:

$$\hbar^n \left\{ \bar{\mathcal{D}}_A^{(0)} \bar{\psi}_i^{(n)} + H_i^{(n)} \right\} = 0,$$

where

$$H_i^{(n)} = \bar{\sigma}^\rho A_\rho^{(n)} \bar{\psi}_i^{(0)} + \sum_{(p;\ l,m) \in I(n)} \frac{1}{p\ !} \left(\bar{\sigma}^\rho A_\rho^{(l)} (\overleftrightarrow{\Delta})^p \bar{\psi}_i^{(m)} \right)$$

Homogeneous part has k zero modes : η_i .
We obtained the following.

Theorem 3. *Let $\bar{\psi} = (\bar{\psi}_i)$ be a zero mode of $\bar{\mathcal{D}}_A\star$ as above. Then*

$$\bar{\psi}_i^{(n)} = \sum_{j=1}^{k} a_{n,i}^j \eta_j - \frac{1}{\mathcal{D}_A^{(0)}\bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H_i^{(n)},$$

$$\eta_j = O(|x|^{-3}), \quad \frac{1}{\mathcal{D}_A^{(0)}\bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H_i^{(n)} = O(|x|^{-5+\epsilon}),$$

and

$$\bar{\psi}_i = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{k} a_{n,i}^j \eta_j\right) \hbar^n + O(|x|^{-5+\epsilon}) \ , \eta_j = O(|x|^{-3}).$$

Theorem 4.

If $\mathrm{Ind}\ \not{D}^0 := \dim\ \ker \mathcal{D}_A^{(0)} - \dim\ \ker \bar{\mathcal{D}}_A^{(0)} = -k,$

then $\mathrm{Ind}\ \not{D}\star := \dim\ \ker \mathcal{D}_A \star - \dim\ \ker \bar{\mathcal{D}}_A \star = -k\ .$

(5) From Instanton to ADHM

Completeness

Let us introduce $\star_x$ as $\star$ associated with variable x.

$$\star_x \bar{\psi}(x)\bar{\psi}^\dagger(y)\star_y = \star_x \delta(x-y)\star_y - \star_x \mathcal{D}_A \star_x G_A(x,y)\star_y \overleftarrow{\mathcal{D}}_A \star_y,$$

where $\Delta_A \star G_A(x,y) = \delta(x-y)$.

Derivation of ADHM equations

$$T^\mu \ := \ \int_{\mathbb{R}^4} d^4x \frac{1}{2}\left(x^\mu \star \bar{\psi}^\dagger \star \bar{\psi} + \bar{\psi}^\dagger \star \bar{\psi} \star x^\mu\right)$$

$$T^\mu T^\nu = \frac{1}{4}\int_{\mathbb{R}^4} d^4x \bar\psi^\dagger \star \bar\psi \star x^\nu \star x^\mu$$

$$-\frac{1}{4}\int_{S^3} dS_x^\rho \int_{\mathbb{R}^4} d^4y (x^\mu \star_x \bar\psi^\dagger(x)\sigma_\rho) \star_x G_A(x,y) \star_y (\bar\sigma^\nu \bar\psi(y)) + \cdots$$

where $dS_x^\mu = |x|^2 x^\mu d\Omega$ and $d\Omega$ is the solid angle.

• Introduce an asymptotically parallel section $g^{-1}S$ of $S^+ \otimes E$ by

$$\bar\psi = -\frac{g^{-1}Sx^\dagger}{|x|^4} + O(|x|^{-4}).$$

Note that $A \to g^{-1} \star dg$, $D_\mu \star g^{-1} \to 0$ at $r \to \infty$.

- Using the asymptotic behavior, 2nd becomes

$$\frac{1}{8}tr(S^{\dagger}S\bar{\sigma}^{\mu}\sigma^{\nu}),$$

where tr is trace with respect to spinor suffixes.

- In the $[T^{\mu},T^{\nu}]^{+}$ combination, 1st becomes $-\theta^{\mu\nu+}$.

  Finally we get ADHM Eqs.

$$[T^{\mu},T^{\nu}]^{+}=\frac{1}{2}tr(S^{\dagger}S\bar{\sigma}^{\mu\nu})-\theta^{\mu\nu+}$$

Completeness and Uniqueness

We can prove the one to one correspondence between the ADHM data and the Instanton solution.

- Instanton $\Rightarrow$ ADHM $\Rightarrow$ Instanton
- ADHM $\Rightarrow$ Instanton $\Rightarrow$ ADHM

(6) NC U(1) Instantons

There is no U(1) instanton in the commutative limit, therefore we change the strategy of formal expansion. We set

$$H(n) := \{f \,|\, ||f|| := \sup_{x\in\mathbb{R}^4} (1+|x|)^{n+\alpha}|\partial_x^\alpha f(x)| < \infty,$$

$$\sup_{x\in\mathbb{R}^4} (1+|x|)^{n+\alpha+1}|\partial_x^\alpha f(x)| = \infty \text{ for any } \alpha \in \mathbb{N}_{\geq 0} \lim\},$$

$$\mathcal{H}(n) := \{f(x) | f(x) = \sum_{k=n}^{\infty} f^{[k]}(x), f^{[n]}(x) \neq 0\}.$$

ex) Formal expand. $A_\mu = \sum_{k=n}^{\infty} A_\mu^{[k]} \in \mathcal{H}(n)$

- $A_\mu \in \mathcal{H}(1)$ case (roughly $A_\mu \sim O(1/|x|)$)

Let us solve $P_{\mu\nu,\rho\tau} F^{\rho\tau} = 0.$ recursively.
The leading and Next leading (2nd and 3rd order)

$$P^{\mu\nu,\rho\tau}(\partial_\rho A_\tau^{[i]} - \partial_\tau A_\rho^{[i]}) = 0. \quad (i = 1, 2)$$

By using an arbitrary scalar field $\phi \in H(0)$, we solve this equation as

$$A_\mu^{[i]} = \partial_\mu \phi^{[i-1]}$$

The 4-th order Eq. is

$$P^{\mu\nu,\rho\tau}(\partial_\rho A_\tau^{[3]} - \partial_\tau A_\rho^{[3]} + i[A_\rho^{[1]}, A_\tau^{[1]}]_{\overleftrightarrow{\Delta}}) = 0,$$

where $[A,B]_{\overleftrightarrow{\Delta}} := A\overleftrightarrow{\Delta}B - B\overleftrightarrow{\Delta}A$. The solution :

$$A^{\mu[3]} = -i\partial_\nu \int_{\mathbb{R}^4} G(x,y)P^{\mu\nu,\rho\tau}[A_\rho^{[1]}, A_\tau^{[1]}]_{\overleftrightarrow{\Delta}}(y)d^4y.$$

$A^{[l]}(l \geq 4)$ is determined by the same way.

But this solution dose not make non-zero instanton number. Because instanton number is unchanged under deformation

$$A_\mu^{[1]} \to A_\mu^{[1]} + O(|x|^{-3}).$$

(7) New NC U(1) Instantons

- $A_\mu \in \mathcal{H}(0)$ case (roughly $A_\mu \sim O(1/\log|x|)$

The 1st order Eq. is given by

$$P^{\mu\nu,\rho\tau}(\partial_\rho A_\tau^{[0]} - \partial_\tau A_\rho^{[0]}) = 0.$$

By using an arbitrary scalar field $\phi \in H(-1)$,

$$A_\mu^{[0]} = \partial_\mu \phi^{[-1]}$$

The next leading (2-th order) Eq. is given by

$$P^{\mu\nu,\rho\tau}(\partial_\rho A_\tau^{[1]} - \partial_\tau A_\rho^{[1]} + i[A_\rho^{[0]}, A_\tau^{[0]}]_{\overleftrightarrow{\Delta}}) = 0.$$

By using the similar way of the previous case, we obtain

$$A^{\mu[1]} = -i \int_{\mathbb{R}^4} \frac{\partial}{\partial y^\nu} G(x,y) P^{\mu\nu,\rho\tau} [A_\rho^{[1]}, A_\tau^{[1]}]_{\overleftrightarrow{\Delta}}(y) d^4 y.$$

$A^{[l]} (l \geq 2)$ is determined recursively.
Thus we obtain non-trivial U(1) instantons.

Let us consider $F = \sum_{k=1}^{\infty} F^{[k]}$ of this instanton.

Note that $F^{[1]} = 0$ since of $A^{[0]}$. $F^{[2]}$ is given as

$$F^{[2]} = \partial_\mu A_\nu^{[1]} - \partial_\nu A_\mu^{[1]} + i[A_\rho^{[0]}, A_\tau^{[0]}]_{\overleftrightarrow{\Delta}}.$$

Therefore instanton number is given by

$$\frac{1}{8\pi^2}\int tr\ F \wedge \star F = \frac{1}{8\pi^2}\int tr\ F^{[2]} \wedge \star F^{[2]}.$$

We rewrite instanton $\sharp$ " $\dfrac{1}{8\pi^2}\displaystyle\int trF \wedge \star F$" as

$$\frac{1}{8\pi^2}\int \ d(A \wedge \star dA + \frac{2}{3}A \wedge \star A \wedge \star A +) + \frac{1}{8\pi^2}\int P_\star$$

where

$$P_\star \ = \ A \wedge \star A \wedge \star A \wedge \star A + \cdots$$

$\int P_\star$ is 0 in the commutative case, but does not vanish in noncommutative space in general.

Example of NC U(1) instanton

For simplicity, we set the NC parameter as

$$\theta = \begin{pmatrix} 0 & h & 0 & 0 \\ -h & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -p & 0 \end{pmatrix},$$

which does not break generality.

We put

$$\phi^{[-1]}(x) = \int_0^{|x|} \frac{1}{\log(e+|x|)} d|x| \ \in H(-1),$$

in other word

$$A_\mu^{[0]} = \frac{x_\mu}{|x|\log(e+|x|)}.$$

Then $A \wedge \star A \wedge \star A \wedge \star A$ is obtained as

$$-\frac{8hp}{\{\log(e+|x|)\}^5 |x|^3 \, (e+|x|)} + O(|x|^{-6}).$$

Its integration over $\mathbb{R}^4$ is done easily as

$$\begin{aligned}
&\int_{\mathbb{R}^4} A \wedge \star A \wedge \star A \wedge \star A \\
&\sim -\int_{\mathbb{R}^4} d^4x \frac{8hp}{\{\log(e+|x|)\}^5 |x|^3 \,(e+|x|)} \\
&= -2hp \times 2\pi^2 = -4\pi^2 hp.
\end{aligned}$$

Note that
the instanton $\sharp$ is deformed by the NC parameter.
This future is different from ADHM instantons.

(8) Deformation of Vortex

Theorem 5. (A_0, ϕ_0) *satisfy the Vortex Eqs. Then there exists a unique solution* (A, ϕ) *of the NC vortex equations with* $A|_{\hbar=0} = A_0$, $\phi|_{\hbar=0} = \phi_0$, *and its vortex number is preserved:*

$$N = N_0 \ , \ i.e. \ \frac{1}{2\pi}\int d^2x \ B = \frac{1}{2\pi}\int d^2x \ B_0 \ .$$

(9) Conclusions

- The Smooth NC Deformation of Instanton exists.
- The Instanton Number is not deformed in $\mathbb{R}^4$.
- The Index theorem is not deformed.
- The Green's function exists.
- The ADHM Eqs. are derived.

1 to 1 of ADHM$\Leftrightarrow$Instanton exists

- The NC U(1) instanton is constructed by deformation quantization
- Instanton $\sharp$ depend on NC parameter.
- The Smooth NC Deformation of Vortex exists and it's Uniquely Determined.
- The Vortex Number is not deformed in $\mathbb{R}^2$.