A Tutorial on Quantum Clifford Algebras and Some of Their Applications!

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Outline

V vector space over $\mathbb{K}$, endowed with $g \in$ lin-Hom$(V \times V, \mathbb{K})$.
$v \in V$, $g(\mathbf{v}, \mathbf{v}) = Q(v)$,
$\mathcal{A}$ unital associative algebra.
$\gamma : V \to \mathcal{A}$ linear.

The pair $(\mathcal{A}, \gamma)$ is a Clifford algebra with respect to (V, g) if $\mathcal{A}$ is generated by $\{\gamma(\mathbf{v}) \mid \mathbf{v} \in V\}$ e $\{a1_{\mathcal{A}} \mid a \in \mathbb{K}\}$ and

$$\boxed{\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2g(\mathbf{u}, \mathbf{v})1_{\mathcal{A}}}$$

In orthonormal basis $\{\mathbf{e}_i\} \in V$

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2g(\mathbf{e}_i, \mathbf{e}_j)\mathbf{1} = 2g_{ij}\mathbf{1}.$$

The definition of the Clifford algebra reads:

$$\mathcal{C}\ell(V, g) \simeq \mathbf{Alg}(\mathbf{e}_i) \bmod \mathbf{e}_i\mathbf{e}_j \ = \ 2g_{ij}\mathbf{1} - \mathbf{e}_j\mathbf{e}_i \,.$$

$$\mathcal{I} = \{X \in T(V) \,|\, X = L \otimes (\mathbf{v} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v})\mathbf{1}) \otimes M, \;\; L, M \in T(V)\}$$

'Square law' of Clifford algebras.

$$\mathcal{C}\ell(V, g) := \frac{T(V)}{\mathcal{I}}$$

Classification

$$\boxed{\mathcal{C}\ell_{p+1,q+1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q}}$$

$$\mathcal{C}\ell_{q+2,p} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q}$$

$$\mathcal{C}\ell_{q,p+2} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{p,q}$$

$$\mathcal{C}\ell_{p,q+8} \simeq \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{0,8} \quad \text{(Atiyah-Bott-Shapiro)}$$

$p-q$ mod 8	0	1	2	3
$\mathcal{C}\ell_{p,q}$	$\mathcal{M}(2^{[n/2]}, \mathbb{R})$	$\mathcal{M}(2^{[n/2]}, \mathbb{R})^{\oplus 2}$	$\mathcal{M}(2^{[n/2]}, \mathbb{R})$	$\mathcal{M}(2^{[n/2]}, \mathbb{C})$
$p-q$ mod 8	4	5	6	7
$\mathcal{C}\ell_{p,q}$	$\mathcal{M}(2^{[n/2]}, \mathbb{H})$	$\mathcal{M}(2^{[n/2]-1}, \mathbb{R})^{\oplus 2}$	$\mathcal{M}(2^{[n/2]-1}, \mathbb{H})$	$\mathcal{M}(2^{[n/2]}, \mathbb{C})$

$\mathcal{C}\ell(V,g) \simeq \mathcal{M}(k \times k, \mathbb{L})$, where $\mathbb{L} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{R} \oplus \mathbb{R}, \mathbb{H} \oplus \mathbb{H}$

Let $B : V \times V \to \mathbb{R}$ a bilinear form, and then $B = g + A$, where $A^T = -A$, and $g^T = g$.

$$\begin{aligned} B(\mathbf{u},\mathbf{v}) &= \mathbf{u} \mathbin{\lrcorner_B} \mathbf{v} \\ A(\mathbf{u},\mathbf{v}) &= \mathbf{u} \mathbin{\lrcorner_A} \mathbf{v} \\ g(\mathbf{u},\mathbf{v}) &= \mathbf{u} \mathbin{\lrcorner_g} \mathbf{v} = \mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

we could define

The pair $(\mathcal{A}, \gamma)$ is a quantum Clifford algebra with respect to (V, B) if $\mathcal{A}$ is generated by $\{\gamma(\mathbf{v}) \mid \mathbf{v} \in V\}$ e $\{a1_{\mathcal{A}} \mid a \in \mathbb{K}\}$ and

$$\boxed{\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2B(\mathbf{u}, \mathbf{v})1_{\mathcal{A}}}$$

it would imply that $B = g$ is a **symmetric** bilinear form !!!

B-dependent Clifford product $\underset{B}{\mathbf{uv}}$ of two 1-vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{C}\ell(B,V)$ can be decomposed in (different ways) into scalar and bi-vector parts as follows

$$\underset{B}{\mathbf{uv}} = \mathbf{u} \underset{g}{\lrcorner} \mathbf{v} + \mathbf{u} \dot{\wedge} \mathbf{v} \qquad \text{Hestenes}$$

$$\underset{B}{\mathbf{uv}} = \mathbf{u} \underset{B}{\lrcorner} \mathbf{v} + \mathbf{u} \wedge \mathbf{v} \qquad \text{Oziewicz, Lounesto, Abłamowicz, Fauser,}$$

where $\mathbf{u} \dot{\wedge} \mathbf{v} = \mathbf{u} \wedge \mathbf{v} + A(\mathbf{u},\mathbf{v}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \lrcorner_A \mathbf{v}$.

Clifford product

Given $\mathbf{u} \in V$ and $\psi \in \Lambda(V)$

$$\mathbf{u}\psi = \mathbf{u} \mathbin{\underset{g}{\lrcorner}} \psi + \mathbf{u} \wedge \psi \quad \text{(Standard Clifford product)}$$

$$\begin{aligned} \underset{B}{\mathbf{u}\psi} &= \mathbf{u} \mathbin{\underset{B}{\lrcorner}} \psi + \mathbf{u} \wedge \psi \quad \text{(arbitrary Clifford product)} \\ \underset{B}{\mathbf{u}\psi} &= \mathbf{u} \mathbin{\underset{g}{\lrcorner}} \psi + \mathbf{u} \dot{\wedge} \psi \end{aligned}$$

where $\mathbf{u} \dot{\wedge} \mathbf{v} = \mathbf{u} \wedge \mathbf{v} + A(\mathbf{u}, \mathbf{v}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \lrcorner_A \mathbf{v}$.

$$\gamma(\mathbf{u})\gamma(\mathbf{v}) + \gamma(\mathbf{v})\gamma(\mathbf{u}) = 2\,g(\mathbf{u},\mathbf{v})\mathbf{1}\,.$$

In the anticommutation relation only the symmetric part of B occurs. However, the anticommutators are altered

$$\gamma(\mathbf{u})\gamma(\mathbf{v}) - \gamma(\mathbf{v})\gamma(\mathbf{u}) = 2\,\mathbf{u}\wedge\mathbf{v} + 2\,A(\mathbf{u},\mathbf{v})\mathbf{1}\,.$$

The Z_n-grading depends directly on the presence of the antisymmetric part. Two different *Graßmann algebras*! One is Z_n-graded while the other is not!

$$\begin{aligned}\mathbf{u}\dot{\wedge}\mathbf{v} &= \mathbf{u}\wedge\mathbf{v} + A(\mathbf{u},\mathbf{v})\\ \mathbf{u}\dot{\wedge}\mathbf{v}\dot{\wedge}\mathbf{w} &= \mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} + A(\mathbf{u},\mathbf{v})\mathbf{w} + A(\mathbf{v},\mathbf{w})\mathbf{u} + A(\mathbf{w},\mathbf{u})\mathbf{v}\\ &\text{etc.}\end{aligned}$$

Dual Space

$$\tau : V \to V^*$$
$$\tau(\mathbf{u})(\mathbf{v}) \quad = \quad \mathbf{u} \underset{B}{\lrcorner} \mathbf{v} = B(\mathbf{u}, \mathbf{v}).$$

Dual isomorphism $\tau : \bigwedge V \to \bigwedge V^*$.

The Clifford algebra $\mathcal{C}\ell(B, V)$ is then constructed: $\gamma_{\mathbf{u}}^{\pm} : \bigwedge V \mapsto \bigwedge V$ for any $\mathbf{u} \in V$ as:

$$\gamma_{\mathbf{u}}^{\pm} := \mathbf{u} \underset{B}{\lrcorner} \cdot \pm \mathbf{u} \wedge \cdot$$

So, the 3 equivalent definitions in $\mathcal{C}\ell(V, g)$ are not equivalent in $\mathcal{C}\ell(V, B)$.

[Ablamowicz, Fauser, 99]

Theorem: $\mathcal{C}\ell(V,B) \cong \mathcal{C}\ell(V,g)$ as $\mathbb{Z}_2$-graded Clifford algebras.

$$e_\wedge^F := \sum \frac{1}{n!} \wedge^n F = \mathbf{1} + F + \frac{1}{2} F \wedge F + \cdots + \frac{1}{n!} \wedge^n F + \cdots$$

Isomorphism ϕ given as:

$$\begin{aligned} \mathcal{C}\ell(V,B) &= \phi^{-1}(\mathcal{C}\ell(V,g)) \\ &= e_\wedge^{-F} \wedge \mathcal{C}\ell(V,g) \wedge e_\wedge^F \\ &\simeq (\mathcal{C}\ell(V,g), <.>_r^A) \end{aligned}$$

where $<.>_r^A$ denotes the A-dependent $\mathbb{Z}_n$-grading $\mathbf{u} \dot{\wedge} \mathbf{v} = \mathbf{u} \wedge \mathbf{v} + A(\mathbf{u},\mathbf{v})$.

$$\begin{aligned} \mathbf{u} \underset{g}{\lrcorner} \cdot &\mapsto \mathbf{u} \underset{B}{\lrcorner} \cdot = \mathbf{u} \underset{g}{\lrcorner} \cdot + (\mathbf{u} \underset{g}{\lrcorner} F) \wedge \cdot \\ \mathbf{u} \wedge \cdot &\mapsto \mathbf{u} \dot{\wedge} \cdot \end{aligned}$$

Sketch

Given $\psi \in \mathcal{C}\ell(V,g)$ and $\mathbf{u} \in V$

$$
\begin{array}{ll}
i) & e_\wedge^{-F} \wedge e_\wedge^{F} = \mathbf{1}, \\
ii) & e_\wedge^{-F} \wedge \mathbf{u} \wedge e_\wedge^{F} \wedge \psi = \mathbf{u} \wedge \psi, \\
iii) & e_\wedge^{-F} \wedge (\mathbf{u} \,\lrcorner_g (e_\wedge^{F} \wedge \psi)) = \mathbf{u} \,\lrcorner_g \psi + (\mathbf{u} \,\lrcorner_g F) \wedge \psi,
\end{array}
$$

Given $\mathbf{u}, \mathbf{v} \in V$, every antisymmetric bilinear form can be written as

$$A(\mathbf{u}, \mathbf{v}) := F \,\lrcorner_g (\mathbf{u} \wedge \mathbf{v})$$

where $F \in \Lambda^2(V)$ is an appropriately chosen.

Wedges are not altered and the new contractions are given by

$$\mathbf{u} \,\lrcorner_B \cdot \equiv \mathbf{u} \,\lrcorner_g \cdot + (\mathbf{u} \,\lrcorner_g F) \wedge \cdot$$

Gradings are mixed. It preserves the parity: $\mathbb{Z}_2$-graded isomorphism.

Let $V_{p,q} = (V, g_{p,q})$, $g = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$.

$$V_{p,q} = N_{p-1,q-1} \perp_g M_{1,1} \quad \textbf{Witt theorem}$$

Clifford map $\gamma : V_{p,q} \mapsto \mathcal{C}\ell_{p,q}$, its natural restrictions $\gamma' : N_{p-1,q-1} \mapsto \mathcal{C}\ell_{p-1,q-1}$, $\gamma'' : M_{1,1} \mapsto \mathcal{C}\ell_{1,1}$, **Periodicity Theorem:** $\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p-1,q-1} \otimes \mathcal{C}\ell_{1,1}$.

Restrictions of the isomorphism $\phi^{-1}|_N$ and $\phi^{-1}|_M$, calculate the decomposition of $\mathcal{C}\ell_{p,q}(B)$:

$$\begin{aligned}
\mathcal{C}\ell_{p,q}(B) &= \phi^{-1}(\mathcal{C}\ell_{p,q}(g)) \\
&= \phi^{-1}\left[\mathcal{C}\ell_{p-1,q-1}(g|_N) \otimes \mathcal{C}\ell_{1,1}(g|_M)\right] \\
&= \mathcal{C}\ell_{p-1,q-1}(B|_N)(\phi^{-1}\otimes)\mathcal{C}\ell_{1,1}(B|_M) \\
&= \mathcal{C}\ell_{p-1,q-1}(B|_N) \otimes_{\phi^{-1}} \mathcal{C}\ell_{1,1}(B|_M).
\end{aligned}$$

Remark

The (deformed) tensor product $\otimes_{\phi^{-1}}$ is not braided by construction: no restrictions on ϕ^{-1}, a priori.

$$(c_{V,W} \otimes I) \circ (I \otimes c_{U,W}) \circ (c_{U,V} \otimes I) \neq (I \otimes c_{U,V}) \circ (c_{U,W} \otimes I) \circ (I \otimes c_{V,W})$$

where

$$c_{U,W} : U \otimes W \to W \otimes V$$

Quantum Clifford algebras *do not* come in general with periodicity theorems as e.g. the famous Atiyah-Bott-Shapiro mod 8 index theorem.

Let $< \mathbf{1}, \mathbf{e}_1, \ldots, \mathbf{e}_{2n} >$ be a set of generators of the Clifford algebra $\mathcal{C}\ell(B, V)$,

$$B_{i,j} := \begin{cases} 0, & \text{if } 1 \leq i,j \leq n \text{ or } n < i,j \leq 2n, \\ q, & \text{if } i = j - n \text{ or } i - 1 - n = j, \\ -(1+q), & \text{if } i + 1 = j - n \text{ or } i = j + 1 - n, \\ -1, & \text{if } |i - j - n| \geq 2 \text{ and } i > n, \\ 1, & \text{otherwise.} \end{cases}$$

For example, when $n = 4$, then

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & q & -1-q & 1 & 1 \\ 0 & 0 & 0 & 0 & -1-q & q & -1-q & 1 \\ 0 & 0 & 0 & 0 & 1 & -1-q & q & -1-q \\ 0 & 0 & 0 & 0 & 1 & 1 & -1-q & q \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ q & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & q & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & q & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hecke Algebras

$$\boxed{b_i := \mathbf{e}_i \wedge \mathbf{e}_{i+n}}$$

$$\begin{aligned} b_i^2 &= (1-q)b_i + q \\ b_i b_j &= b_j b_i, \quad |i-j| > 1, \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}. \end{aligned}$$

Hecke algebras !

Conformal Maps

For $n = p + q \leq 5$,

$$\mathrm{Spin}(p,q) = \{\phi \in \mathcal{C}\ell^{+}_{p,q} \mid \tilde{\phi}\phi = 1\}$$
$$\mathrm{Spin}(2,4) = \{\phi \in \mathcal{C}\ell_{4,1} \mid \tilde{\phi}\phi = 1\}$$

Conformal Maps

$$\psi \mapsto (*)\psi\widehat{(*)}^{-1}, \quad \psi \in \mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{3,0} \simeq \begin{pmatrix} \mathcal{C}\ell_{3,0} & \mathcal{C}\ell_{3,0} \\ \mathcal{C}\ell_{3,0} & \mathcal{C}\ell_{3,0} \end{pmatrix}$$

Isomorphism $\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{M}(4, \mathbb{C})$

Conformal Map	.	Matrix of \$pin$_+(2,4)$
Translation	$x \mapsto x + h, \quad h \in \mathbb{R} \oplus \mathbb{R}^3$	$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$
Dilatation	$x \mapsto \rho x, \ \ \rho \in \mathbb{R}$	$\begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1/\sqrt{\rho} \end{pmatrix}$
Rotation	$x \mapsto gx\hat{g}^{-1}, \ \ g \in \$\mathrm{pin}_+(1,3)$	$\begin{pmatrix} g & 0 \\ 0 & \hat{g} \end{pmatrix}$
Invertion	$x \mapsto -\overline{x}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Transvection	$x \mapsto x + x(hx+1)^{-1}, \quad h \in \mathbb{R} \oplus \mathbb{R}^3$	$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$

Generators of Conf(1,3) are expressed from the $\{\gamma_\mu\} \in \mathcal{C}\ell_{1,3}$ as

$$P_\mu = \frac{1}{2}(\gamma_\mu + i\gamma_\mu\gamma_5), \qquad K_\mu = -\frac{1}{2}(\gamma_\mu - i\gamma_\mu\gamma_5), \qquad D = \frac{1}{2}i\gamma_5, \qquad M_{\mu\nu} = \frac{1}{2}(\gamma_\nu \wedge \gamma_\mu). \tag{1}$$

They satisfy the following relations:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \qquad [K_\mu, K_\nu] = 0, \qquad [M_{\mu\nu}, D] = 0,\\
[M_{\mu\nu}, P_\lambda] &= -(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu),\\
[M_{\mu\nu}, K_\lambda] &= -(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu),\\
[M_{\mu\nu}, M_{\sigma\rho}] &= g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma},\\
[P_\mu, K_\nu] &= 2(g_{\mu\nu}D - M_{\mu\nu}), \qquad [P_\mu, D] = P_\mu, \qquad [K_\mu, D] = -K_\mu.
\end{aligned}$$

$\mathcal{C}\ell(V,B) = e^F\ \mathcal{C}\ell(V,g)\ e^{-F}$

κ-Poincaré algebraic sector is presented by the following commutation relations:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 = [M_{ij}, P_0], \qquad [\epsilon_{ijk} M_{ij}, P_\ell] = i\,\epsilon_{i\ell r} P_r \\
[K_3, P_0] &= \frac{i}{2}\gamma_3(1+i\gamma_5), \qquad [K_3, P_2] = \frac{i}{2\kappa}\gamma_2\gamma_3(1+i\gamma_5), \qquad [P_3, K_3] = \frac{i}{2\kappa}(1+i\gamma_5) - i\kappa \sinh\Big(\frac{\gamma \\
[K_3, P_1] &= \frac{i}{2\kappa}\gamma_3\gamma_1(1+i\gamma_5), \qquad [K_\pm, P_0] = \frac{1}{2}(\mp\gamma_2 + i\gamma_1)(1+i\gamma_5), \\
[K_\pm, P_2] &= \mp i\kappa \sinh\left(\frac{\gamma_0 + i\gamma_0\gamma_5}{\kappa}\right) \pm \frac{1}{2\kappa}\gamma_3(1+i\gamma_5) \\
[K_\pm, P_1] &= i\kappa \sinh\left(\frac{\gamma_0 + i\gamma_0\gamma_5}{\kappa}\right) - \frac{i}{2\kappa}\gamma_3(1+i\gamma_5), \qquad [K_\pm, P_3] = \mp\frac{1}{2\kappa}\gamma_3(\gamma_2 \mp i\gamma_1)(1 \pm i\gamma_5) \\
[M_+, M_-] &= \frac{1}{2}\gamma_1 \wedge \gamma_2, \qquad [M_{12}, M_\pm] = \pm\frac{1}{2}\gamma_3(\gamma_1 \pm i\gamma_2) \\
[K_+, K_-] &= -\gamma_1 \wedge \gamma_2 \cosh\left(\frac{\gamma_0 + i\gamma_0\gamma_5}{\kappa}\right) - \sinh\left(\frac{\gamma_0 + i\gamma_0\gamma_5}{\kappa}\right) \\
[K_\pm, K_3] &= \pm 1 \pm \frac{\gamma_0}{4\kappa}(1+i\gamma_5)\gamma_3(\gamma_2 - i\gamma_1) + \frac{1}{8\kappa}\left((i+1)\gamma_3 \wedge \gamma_0(\gamma_2 + i\gamma_1)(1+i\gamma_5)\right) \\
[M_\pm, K_\pm] &= \mp\frac{1}{8\kappa}\gamma_3(1+i\gamma_1\gamma_2)(1\mp 1)(1+i\gamma_5), \qquad [M_{12}, K_3] = 0, \qquad [M_{12}, K_\pm] = \mp\frac{1}{2}(\gamma_1 \\
[M_\pm, K_\mp] &= \left(\mp\gamma_3 + \frac{i}{8\kappa}(1\mp 1)(1-\gamma_1\gamma_2)\wedge\gamma_3\right)(1-i\gamma_5) + \frac{1}{4}(\mp\gamma_1\wedge\gamma_2 \pm 2)\gamma_3(1+i\gamma_5) \\
[M_\pm, K_3] &= \mp\frac{1}{2}(\gamma_1 \pm i\gamma_2)\wedge\gamma_0 \pm \frac{1}{8\kappa}(\gamma_1\wedge\gamma_2)(\gamma_1 + i\gamma_2)(1+i\gamma_5) + \frac{i}{4\kappa}(\gamma_2 \mp i\gamma_1)(1+i\gamma_5).
\end{aligned}
$$

Hopf sector the Clifford-Hopf algebra

- Unit map $\eta : \mathbb{K} \to \mathcal{C}\ell(V, g)$.
- Co-product $\Delta : \mathcal{C}\ell(V, g) \to \mathcal{C}\ell(V, g) \otimes \mathcal{C}\ell(V, g)$
- Co-unit $\epsilon : \mathcal{C}\ell(V, g) \to \mathbb{K}$
- Compatibility of the algebra and the co-algebra structure:
  $\epsilon(\mathrm{Id}) = 1$, $\epsilon(\psi\phi) = \epsilon(\psi)\epsilon(\phi)$

  $$\Delta(\mathrm{Id}) = \mathrm{Id} \otimes \mathrm{Id}, \qquad \Delta(\psi) = \psi \otimes \mathrm{Id} + \mathrm{Id} \otimes \psi, \qquad \Delta(\psi\phi) = \Delta(\psi)\Delta(\phi).$$

  Define also the antipode $S : \Lambda^p(V) \to \Lambda^p(V)$ as $S(\psi_p) = (-1)^p \psi_p$, where $\psi_p \in \Lambda^p(V) \subset \Lambda(V)$.
- antipode = graded involution.
- $(\Lambda(V), g, \gamma, \epsilon, \Delta, S)$ is is the Clifford-Hopf algebra.
- All axioms for a Hopf algebra are satisfied, together with the antipode axioms:

  $$\wedge \circ (S \otimes \mathrm{Id}) = \eta \circ \epsilon, \qquad \wedge \circ (\mathrm{Id} \otimes S) = \eta \circ \epsilon$$
- Since S^2 = Id, the Clifford-Hopf algebra is $\mathbb{Z}_2$-graded co-commutative.

Conformal Maps

- κ-Poincaré algebra obtained via $\mathcal{U}_q(\mathfrak{so}(3,2))$ + quantum de-Sitter contraction.
- quantum AC $\mapsto$ deformed anti-de Sitter algebra $\mathcal{U}_q(\mathfrak{so}(3,2))$.
- $SU(2,2) \simeq \mathrm{Spin}_+(2,4) = \mathrm{Spin}_+(2,4)(g)$, and by the Wick isomorphism $\phi(\mathrm{Spin}_+(2,4)) = Spin(2,4)(B)$, the κ-Poincaré algebra and also the Hopf-algebraic structure are obtained...
- DSR

...in full compliance with

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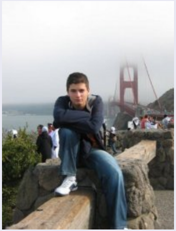
♡ Dedicated to My Beloved Family ♡

♡  ♡  ♡  ♡