Deforming SW curve

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Foreword

- Mainly based on the work arXiv:1006.4822
- Results: A system of Bethe-Ansatz type equations, which specify a unique array of Young tableau responsible for the leading contribution to the Nekrasov partition function in the $\epsilon_2 \to 0$ limit is derived.
- The prepotential with generic ϵ_1 is directly related to the (rescaled by ϵ_2) number of total boxes of these Young tableau. Moreover, all the expectation values of the chiral fields $\langle \mathrm{tr}\phi^J \rangle$ are simple symmetric functions of their column lengths.
- An entire function whose zeros are determined by the column lengths is introduced. It satisfies a functional equation, resembling Baxter's equation in 2d integrable models. This functional relation directly leads to a nice generalization of the equation defining Seiberg-Witten curve.

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Plan

- ϵ deformation and Nekrasovs partition function
- Prepotential in the limit $\epsilon_2 \to 0$
- The functional equation
- Deformed Seiberg-Witten curve
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- Explicit solution for $U(1)$
- Conclusions

Generalized partition function of $\mathcal{N} = 2$ SYM

The idea of considering $\mathcal{N} = 2$ SYM theories in a specific presently commonly known as Ω- background is proven to be extremely fruitful. The general Ω- background is characterized by two parameters ϵ_1, ϵ_2 introduced in [Moor,Nekrasov,Shatashvili 'arXiv:hep-th/9712241], Losev,Nekrasov,Shatashvili 'arXiv:hep-th/9801061 to regularize the integrals over moduli space of instantons. In arXiv:hep-th/0206161 Nekrasov showed how the partition function in this background is related to the Seiberg-Witten prepotential. In the same paper he performed explicit calculation of the prepotential up to 5 instantons choosing $h = \epsilon_1 = -\epsilon_2$ and demonstrated that at vanishing h one exactly recovers the results extracted from the Seiberg-Witten curve.

- In Flume, R.P. 'arXiv:hep-th/0208176 a closed combinatorial formula which allows to calculate the Nekrasov partition function for generic ϵ_1, ϵ_2 was found. The partition function is represented as a sum over arrays of Young tableau with total number of boxes equal to the number of instantons.
- The partition function with generic ϵ_1, ϵ_2 is essential also from the point of view of the recently established AGT duality Alday,Gaiotto,Tachikawa ' arXiv:0906.3219 relating this partition function to the conformal blocks in 2d Conformal Field Theory.

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In a parallel very interesting development Nekrasov and Shatashvily in 'arXiv:0908.4052 show that when $\epsilon_2 = 0$ the prepotential is related to the quantum integrable many body systems. This case is also the main subject of consideration of the present talk.

Note one more point which to my opinion makes the investigation of $\epsilon_2 = 0$ case even more interesting: namely, due to above mentioned AGT relation this should be related to the quasi-classical ($c \to \infty$) limit of conformal blocks and hence to the classical Liouville field theory.

Thus it is useful to generalize the Seiberg-Witten prepotential including into the game besides unbroken gauge transformations also the space time rotations which allowed to localize instanton contributions around finite number of fixed points.
The general formula giving the contribution of any fixed point has been found [R.Flume, R.P. '02]. For the gauge group $U(N)$ the fixed points are in 1-1 correspondence with the arrays of Young tableau $\vec{Y} = (Y_1, ..., Y_N)$ with total number of boxes $|\vec{Y}|$ being equal to the instanton charge k.

The (holomorphic) tangent space of the moduli space of instantons decomposes into sum of (complex) one dimensional irreducible representations of the Cartan subgroup of $U(N) \times O(4)$ [R.Flume, R.P. '02]

$$\chi = \sum_{\alpha,\beta=1}^{N} e_\beta e_\alpha^{-1} \left\{ \sum_{s \in Y_\alpha} \left(T_1^{-l_{Y_\beta}(s)} T_2^{a_{Y_\alpha}(s)+1} \right) + \sum_{s \in Y_\beta} \left(T_1^{l_{Y_\alpha}(s)+1} T_2^{-a_{Y_\beta}(s)} \right) \right\},$$

where $(e_1, ..., e_N) = (e^{ia_1}, \ldots, e^{ia_N}) \in U(1)^N \subset U(N)$ and $(T_1, T_2) = (e^{i\epsilon_1}, e^{i\epsilon_2}) \in U(1)^2 \subset O(4)$, $a_Y(s)$ ($l_{Y_\alpha}(s)$) is the distance of the right edge of the box s from the limiting polygonal curve of the Young tableaux Y in horizontal (vertical) direction taken with the sign plus if the box $s \in Y_\alpha$ and with the sign minus otherwise.

One-dimensional subgroups of the above mentioned $N+2$ dimensional torus are generated by the vector fields parametrized by $a_1, \ldots, a_N$ and ϵ_1, ϵ_2. From the physical point of view a_α are the vacuum expectation values of the complex scalar of the $\mathcal{N}=2$ gauge multiplet and ϵ_1, ϵ_2 specify a particular gravitational background now commonly called Ω-background. The contribution of a fixed point to the Nekrasov partition function in the basic $\mathcal{N}=2$ case without extra hypermultiplets is simply the inverse determinant of the above mentioned vector field action on the tangent space at given fixed point. All the eigenvalues of this vector field can be directly read off from the character formula.

The result is

$$P_{gauge}(\vec{Y}) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} \frac{1}{E_{\alpha,\beta}(s)(\epsilon - E_{\alpha,\beta}(s))},$$

where

$$E_{\alpha,\beta} = a_\beta - a_\alpha - \epsilon_1 l_{Y_\beta}(s) + \epsilon_2 (a_{Y_\alpha}(s) + 1)$$

In general the theory may include "matter" hypermultiplets in various representations of the gauge group. In that case one should multiply the gauge multiplet contribution by another factor P_{matter}. Here we will consider the case of several (up to four) hypermultiplets in anti-fundamental representation and also the theory with an adjoint hypermultiplet (so called $\mathcal{N} = 2^*$).

The respective matter factors read [Bruzzo, Fucito, Morales, Tanzini '03]

$$P_{antifund}(\vec{Y}) = \prod_{l=1}^{f} \prod_{\alpha=1}^{N} \prod_{s_\alpha \in Y_\alpha} (\chi_{\alpha,s_\alpha} + m_l)$$

$$P_{adj}(\vec{Y}) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} (E_{\alpha,\beta}(s) - M)(\epsilon - E_{\alpha,\beta}(s) - M),$$

where m_l, M are the masses of the hypermultiplets, $\epsilon = \epsilon_1 + \epsilon_2$,

$$\chi_{\alpha,s_\alpha} = a_\alpha + (i_{s_\alpha} - 1)\epsilon_1 + (j_{s_\alpha} - 1)\epsilon_2$$

and i_{s_α}, j_{s_α} are the numbers of the column and the row of the tableaux Y_α where the box s_α is located. Note, to get a fundamental hypermultiplet instead of an antifundamental, one should replace m_l by $\epsilon - m_l$.

The instanton part of Nekrasov partition function is (I use notation $q = e^{2\pi i \tau_g}$ with τ_g the usual gauge theory coupling):

$$Z_{inst} = \sum_{\vec{Y}} q^{|\vec{Y}|} P_{gauge}(\vec{Y}) P_{matter}(\vec{Y})$$

Prepotential in the limit $\epsilon_2 \to 0$

The aim is to derive saddle point equations which determine the $\epsilon_2 = 0$ limit of the deformed prepotential

$$W(a, m, \epsilon_1, q) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z_{inst}(a, m, \epsilon_1, \epsilon_2, q),$$

where a collectively denotes all VEV's of the gauge multiplet and m the masses of possible extra matter hypermultiplets. The instanton part of the partition function is

$$Z_{inst} = 1 + \sum_{k=1}^{\infty} Z_k q^k$$

Two equivalent representations of the Partition function as contour integral LNS 98 and as sum over Young tableau Nek '02,FP '02 are both useful.
The integral representation for the k instanton, $U(N)$ gauge group and f ($f \leq 4$) fundamental hypers:

$$Z_k = \int \prod_{I=1}^{k} \frac{dx_I}{2\pi i} \chi_k(x_I)$$

where

$$\chi_k(x_I) = \frac{1}{k!} \prod_{I,J=1}^{k}{}' \frac{(x_I - x_J)(x_I - x_J + \epsilon_1 + \epsilon_2)}{(x_I - x_J + \epsilon_1)(x_I - x_J + \epsilon_2)}$$

$$\times \prod_{I=1}^{k} \frac{\prod_{a=1}^{f}(x_I + m_\ell)}{\prod_{u=1}^{N}(x_I - a_u)(-x_I + a_u - \epsilon_1 - \epsilon_2)}$$

Following the ideology of Nekrasov,Okounkov arXiv:hep-th/0306238 one expects in the limit $\epsilon_2 \to 0$ that the main contribution to the Z_{inst} will be dominated by certain pole with large $k \sim 1/\epsilon_2$. The poles are in one to one correspondence with the arrays of N Young tableau $Y_1, \dots, Y_N$ which appear in combinatorial representation. Arrange the variables x_I over the k boxes. The variables x_I at a pole are the eigenvalues of the instanton group $U(k)$. The rule: assign to the N corner boxes the expectation values $a_1, \dots, a_N$, increase the value by ϵ_1 (ϵ_2) each time when passing to the next box in horizontal (vertical) direction. Thus the entry of the box $s = (i, j)$, $s \in Y_u$ is

$$x_{u,i,j} = a_u + (i-1)\epsilon_1 + (j-1)\epsilon_2$$

Let us estimate $\log\left(\chi_k q^k\right)$ for a very large $k \sim 1/\epsilon_2$:

$$\log(\chi_k q^k) \sim k \log\, q + \epsilon_2 \sum_{I,J=1}^{k} \left(\frac{1}{x_I - x_J + \epsilon_1} - \frac{1}{x_I - x_J} \right)$$

$$- \sum_{I=1}^{k} \left(\sum_{u=1}^{N} \log\left((x_I - a_u)(-x_I + a_u - \epsilon_1)\right) - \sum_{\ell=1}^{f} \log(x_I + m_\ell) \right)$$

Next insert the values of x_I and replace the discrete sums over those indices $\sim$ small quantity ϵ_2 by integrals: the number of boxes in vertical (ϵ_2) direction is very large, but this number multiplied by ϵ_2 is expected to be finite and will be denoted as $\lambda_{u,i}$.

The calculation is elementary and leads to the conclusion that $\log(\chi_k q^k) \sim \mathcal{H}/\epsilon_2$, where (below the indices $u, v \in 1, \ldots, N$; $\ell \in 1, \ldots, f$; $i, j \in 1, 2, \ldots$):

$$\mathcal{H}(x_{u,i}|\epsilon_1) = \sum_{u,i;v,j} [-G(x_{u,i} - x_{v,j} + \epsilon_1)$$
$$+G(x_{u,i} - x^0_{v,j} + \epsilon_1) + G(x^0_{u,i} - x_{v,j} + \epsilon_1) - G(x^0_{u,i} - x^0_{v,j} + \epsilon_1)]$$
$$+\sum_{u,i;v} [-G(x_{u,i} - a_v) + G(x^0_{u,i} - a_v) - G(x_{u,i} - a_v + \epsilon_1)$$
$$+G(x^0_{u,i} - a_v + \epsilon_1)] + \sum_{u,i;\ell} [G(x_{u,i} + m_\ell) - G(x^0_{u,i} + m_\ell)]$$
$$+\sum_{u,i} (x_{u,i} - (i-1)\epsilon_1 - a_u) \log q$$

where

$$G(x) = x(\log|x| - 1);\ x_{u,i} = a_u + (i-1)\epsilon_1 + \lambda_{u,i};\ x^0_{u,i} = a_u + (i-1)\epsilon_1$$

It is useful to regularize $\mathcal{H}$ assuming that there is an integer L such that the (scaled) lengths of columns $\lambda_{u,i} = 0$ when $i > L$. A nice feature: $\mathcal{H}$ does not depend on the upper limit of the summation indices i, j provided this upper limit is more or equal to L and the column lengths, extremizing the "action" $\mathcal{H}$ are of order $\lambda_{u,i} \sim \mathcal{O}(q^i)$. The extremality condition:

$$-q \prod_{v,j}^{N,L} \frac{(x_{u,i} - x_{v,j} - \epsilon_1)(x_{u,i} - x^0_{v,j} + \epsilon_1)}{(x_{u,i} - x_{v,j} + \epsilon_1)(x_{u,i} - x^0_{v,j} - \epsilon_1)} \frac{\prod_{\ell=1}^{f}(x_{u,i} + m_\ell)}{\prod_{v=1}^{N}(x_{u,i} - a_v + \epsilon_1)(x_{u,i} - a_v)} = 1$$

which closely resembles the Bethe-Ansatz equations of integrable models.

Instanton expansion

A careful analysis shows that the structure of BA-type equations is consistent with $\lambda_{u,i} \sim \mathcal{O}(q^i)$. Having an i-th order solution with this property the equations with $L = i + 1$ uniquely determine not only $\lambda_{u,1}, \ldots, \lambda_{u,i}$ up to the next order $i + 1$ but also in leading order the length of the next column $\lambda_{u,i+1}$, which automatically turns out to be of order $\mathcal{O}(q^{i+1})$. In other words the i-th columns do not contribute up to $(i - 1)$-instanton order. Thus one can start with just $L = 1$ and solve the equation step by step up to desired order. At each stage the problem boils down to a system of linear equations.

$L = 1$ case is simple. Here is the result

$$\lambda_{u,1} = \frac{-q \prod_{\ell=1}^{f}(a_1 + m_\ell)}{\prod_{v \neq u}^{N}(a_u - a_v)(a_u - a_v + \epsilon_1)} + \mathcal{O}(q^2)$$

Summing this expression over u one gets the correct 1-instanton prepotential (with arbitrary ϵ_1 but $\epsilon_2 = 0$). Other non-trivial checks can be performed against known results for $\langle \mathrm{tr}\, \phi^J \rangle$. I have performed higher instanton order computations with various specific choices of N and f always finding perfect agreement with known results.

As an example below is given 2-instanton result for the case with $U(2)$ gauge group without extra hypermultiplets:

$$\lambda_{1,1} = \frac{-q}{2a\epsilon_1(2a+\epsilon_1)} - \frac{\left(-8a^5+4a^4\epsilon_1+22a^3\epsilon_1^2+3a^2\epsilon_1^3+3a\epsilon_1^4+\epsilon_1^5\right)q^2}{8a^3\epsilon_1^3(2a-\epsilon_1)^2(a+\epsilon_1)(2a+\epsilon_1)^3} + \mathcal{O}(q^3)$$

$$\lambda_{1,2} = \frac{-q^2}{8a\epsilon_1^3(a+\epsilon_1)(2a+\epsilon_1)^2} + \mathcal{O}(q^3)$$

$$\lambda_{2,1} = \frac{-q}{2a\epsilon_1(2a-\epsilon_1)} + \frac{\left(8a^5+4a^4\epsilon_1-22a^3\epsilon_1^2+3a^2\epsilon_1^3-3a\epsilon_1^4+\epsilon_1^5\right)q^2}{8a^3(a-\epsilon_1)(2a-\epsilon_1)^3\epsilon_1^3(2a+\epsilon_1)^2} + \mathcal{O}(q^3)$$

$$\lambda_{2,2} = -\frac{q^2}{8a(a-\epsilon_1)\epsilon_1^3(-2a+\epsilon_1)^2} + \mathcal{O}(q^3)$$

The functional equation

For infinitely large large L it is useful to introduce the function

$$Y(z) = \prod_{u=1}^{N} e^{\frac{z}{\epsilon_1}\psi\left(\frac{a_u}{\epsilon_1}\right)} \prod_{i=1}^{\infty} \left(1 - \frac{z}{x_{u,i}}\right) e^{z/x_{u,i}^0}$$

where

$$\psi(x) = \partial_z \log \Gamma(z)$$

If the column lengths tend to zero or $x_{u,i} \to x_{u,i}^0$ at large i fast enough $Y(z)$ is an entire function of z with zeros at $x_{u,i}$. In extreme case when all column lengths are zero one gets

$$Y_0(z) = \prod_{u=1}^{N} \frac{\Gamma\left(\frac{a_u}{\epsilon_1}\right)}{\Gamma\left(\frac{a_u - z}{\epsilon_1}\right)},$$

whose zeros are located at $x_{u,i}^0$.

In view of above definitions the large L "Bethe Ansatz" equations can be represented as

$$-\frac{q}{\epsilon_1^{2N}}\frac{Y(x_{u,i}-\epsilon_1)}{Y(x_{u,i}+\epsilon_1)}\prod_{\ell=1}^{f}(x_{u,i}+m_\ell)=1$$

Denote

$$Q_f(z)=\prod_{\ell=1}^{f}(z+m_\ell)$$

and consider the function

$$(-1/\epsilon_1)^N P_N(z)=\frac{Y(z+\epsilon_1)+\frac{q}{\epsilon_1^{2N}}Q_f(z)Y(z-\epsilon_1)}{Y(z)}.$$

It does not have singularities at finite part of the complex plane since the potential poles $z = x_{u,i}$ are cancelled due to the functional equation. The behaviour at large z is also easy to estimate. At large z the ratio

$$\frac{Y(z+\epsilon_1)}{Y(z)} \sim \frac{Y_0(z+\epsilon_1)}{Y_0(z)} = (-z/\epsilon_1)^N + \mathcal{O}(z^{N-1}).$$

Thus the function $P_N(z)$ is in fact an N-th order polynomial (provided $f \leq 2N$)

$$P_N(z) = z^N + \mathcal{O}(z^{N-1})$$

for $f = 1, 2, \ldots, 2N-1$ and

$$P_N(z) = (1+q)z^N + \mathcal{O}(z^{N-1})$$

for the conformal case $f = 2N$.

Baxters T-Q equation

So we finally arrive at the following functional equation for Y:

$$Y(z+\epsilon_1)+\frac{q}{\epsilon_1^{2N}}Y(z-\epsilon_1)\prod_{\ell=1}^{f}(z+m_\ell)=(-1/\epsilon_1)^N P_N(z)Y(z).$$

which very much resembles the Baxter's $T-Q$ equation well known in the context of 2d integrable statistical systems.

Deformed Seiberg-Witten curve

Denote

$$w(z) = \frac{q}{(-\epsilon_1)^N} \frac{Y(z)}{Y(z+\epsilon_1)}$$

and rewrite the functional equation in the following suggestive form

$$Q_f(z)w(z)w(z-\epsilon_1) - P_N(z)w(z) + q = 0.$$

This equation supplemented with the large z asymptotic condition $w(z) = 1/z^N + \mathcal{O}(1/z^{N+1})$ generalizes the algebraic equation defining the SW curve to the case with finite ϵ_1. The deformation is surprisingly simple. The only difference from the standard case is the shift of one of the arguments by ϵ_1. Putting $\epsilon_1 = 0$ and absorbing the polynomial $Q_f(z)$ by means of redefinition $\sqrt{Q_f(z)}w(z) \to w(z)$ one gets the standard curve equation. Of course, this equation no longer defines a curve in a usual sense and its geometric interpretation needs to be clarified yet.

The Prepotential

The prepotential $W(a, m, \epsilon_1)$ should be equal to the critical value of the "action" $\mathcal{H}$. To evaluate this critical value it is more convenient first to calculate its derivative with respect to the instanton parameter q:

$$q\partial_q W(a, m, \epsilon_1, q) = \sum_{u,i}(x_{u,i} - (i-1)\epsilon_1 - a_u) \equiv \sum_{u,i} \lambda_{u,i}$$

i.e. the $q\partial_q W(a, m, q)$ is simply the sum of all (rescaled) column lengths of the "critical" Young tableau! It is instructive to express this quantity in terms of the functions $Y(z)$, $Y_0(z)$ introduced before:

$$q\partial_q W(a, m, q) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} z\partial_z \log \frac{Y(z)}{Y_0(z)},$$

where the integration contour $\mathcal{C}$ encloses all zeros of $Y(z)$ and $Y_0(z)$ *i.e.* all the points $x_{u,i}$, $x^0_{u,i}$.

Expectation values and the chiral ring

The correlators $\langle tr\, \phi^J \rangle$ constitute the so called chiral ring. In 4 d $\mathcal{N} = 2$ SYM these correlators can be represented as Losev, Nekrasov, Okounkov ' arXiv:hep-th/0302191; Nekrasov, Okounkov 'arXiv:hep-th/0306238; Flume, Fucito, Morales, R.P. 'arXiv:hep- th/0403057

$$\langle tr\, \phi^J \rangle = \langle tr\, \phi^J \rangle_{cl} + \frac{1}{Z_{inst}} \sum_k q^k \int \prod_{I=1}^{k} \frac{dx_I}{2\pi i} \chi_k(x_I)\, O_J(\{x_I\})$$

where the classical part of the expectation value

$$\langle tr\, \phi^J \rangle_{cl} = \sum_{u=1}^{N} a_u^J,$$

Z_{inst} is the instanton partition function and

$$O_J(x_I) = -\sum_{I=1}^{k} \left[(x_I + \epsilon_1 + \epsilon_2)^J - (x_I + \epsilon_1)^J - (x_I + \epsilon_2)^J - x_I^J \right].$$

In the small ϵ_2 limit of our interest $O_J(x_I)$ becomes

$$\lim_{\epsilon_2 \to 0} \epsilon_2 O_J(x_I) = -\sum_{u,i} \left[(x_{u,i} + \epsilon_1)^J - (x^0_{u,i} + \epsilon_1)^J - x^J_{u,i} + {x^0_{u,i}}^J \right].$$

Similar to the case of prepotential the saddle point approximation amounts to keeping one "critical" term. The factors $1/Z_{inst}$ and χ cancel out and we get

$$\langle \mathrm{tr}\, \phi^J \rangle = \sum_{u=1}^{N} a_u^J - \sum_{u,i} \left[(x_{u,i} + \epsilon_1)^J - (x^0_{u,i} + \epsilon_1)^J - x^J_{u,i} + {x^0_{u,i}}^J \right].$$

Recall now the definitions of our functions $Y(z)$, $Y_0(z)$ to rewrite the above expression in following three equivalent ways:

$$\langle \mathrm{tr}\,\phi^J \rangle = \sum_{u=1}^{N} a_u^J - \oint_{\mathcal{C}} \frac{dz}{2\pi i} z^J \partial_z \left(\log \frac{Y(z-\epsilon_1)}{Y_0(z-\epsilon_1)} - \log \frac{Y(z)}{Y_0(z)} \right)$$
$$= \sum_{u=1}^{N} a_u^J - \oint_{\mathcal{C}} \frac{dz}{2\pi i} ((z+\epsilon_1)^J - z^J) \partial_z \log \frac{Y(z)}{Y_0(z)}$$
$$= - \oint_{\mathcal{C}} \frac{dz}{2\pi i} z^J \partial_z \log \frac{Y(z-\epsilon_1)}{Y(z)}$$

The second representation specified to $J = 2$ one gives the well known Matone relation Matone 'arXiv:hep-th/9506102 between the prepotential and $\langle \mathrm{tr}\phi^2 \rangle$ which holds for generic ϵ_1, ϵ_2 as well Flume, Fucito, Morales, R.P. 'arXiv:hep- th/0403057 .

The last representation is also very interesting, it provides a physical interpretation for the function $w(z)$ entering in expression of the deformed SW curve

$$\langle \mathrm{tr}\, \phi^J \rangle = -\oint_{\mathcal{C}} \frac{dz}{2\pi i} z^J \partial_z \log w(z - \epsilon_1)$$

which besides the shift by ϵ_1 coincides with the standard non-deformed expression. Thus $\partial_z \log w(z - \epsilon_1)$ is the analogue of the SW differential. It is worth noting that the "classical" expectation value a_u also can be represented in a similar way

$$a_u = -\oint_{\mathcal{C}_u} \frac{dz}{2\pi i} z^J \partial_z \log w(z - \epsilon_1), \qquad (0.1)$$

where the contour $\mathcal{C}_u$ encloses only the points $x_{u,i}$, $x^0_{u,i}$ with $i = 1, 2, \ldots$ and fixed u. Evidently this is the analogue of the A-cycle integral of the Seiberg-Witten theory.

Explicit solution for $U(1)$

The simplest case with gauge group $U(1)$ without hyper-multiplets can be analysed in full details. The deformed SW curve is now defined as

$$w(z)w(z-\epsilon_1)-(z-c)w(z)+q=0,$$

where c is a constant to be identified later. It is convenient to cast this equation into the form

$$f(x)f(x+1)-xf(x)-t=0.$$

The dictionary is

$$w(z)=-\frac{1}{\epsilon_1}f(-\frac{z-c}{\epsilon_1});\quad t=-\frac{q}{\epsilon_1^2}$$

It is easy to see that the following continued fraction is a solution of the functional equation for $f(x)$:

$$-\frac{f(x)}{t} = \frac{1}{x + \frac{t}{x+1+\frac{t}{x+2+\frac{t}{x+3+\cdots}}}}$$

Gauss has investigated this continued fraction almost two centuries ago. The answer is given by the ratio of (generalized) hyper-geometric functions

$$-\frac{f(x)}{t} = \frac{1}{x}\,\frac{{}_0F_1(x+1,t)}{{}_0F_1(x,t)},$$

where the function ${}_0F_1$ is defined by the power series

$${}_0F_1(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{x(x+1)\cdots(x+k-1)k!}$$

Thus

$$w(z) = \frac{q}{z-c} \frac{{}_0F_1(\frac{\epsilon_1+c-z}{\epsilon_1}, -\frac{q}{\epsilon_1^2})}{{}_0F_1(\frac{c-z}{\epsilon_1}, -\frac{q}{\epsilon_1^2})},$$

At large x the function ${}_0F_1(x,t) \sim \exp(t/x)$. Hence at large z

$$w(z) \sim \frac{1+\mathcal{O}(z^2)}{z-c}$$

In our case there is no singularity outside of the integration contour so it can be freely deformed to a circle of a very large radius. Taking into account the asymptotic behaviour of $w(z)$ we get

$$\langle\phi\rangle \equiv a = c + \epsilon_1.$$

So, the final answer is

$$w(z) = \frac{q}{z-a+\epsilon_1} \frac{{}_0F_1(\frac{a-z}{\epsilon_1}, -\frac{q}{\epsilon_1^2})}{{}_0F_1(\frac{a-z-\epsilon_1}{\epsilon_1}, -\frac{q}{\epsilon_1^2})}.$$

For the sake of completeness let me present here also a closed expression for the entire function $Y(z)$ entering in the definition of $w(z)$ and satisfying the appropriate functional equation:

$$Y(z) = \frac{\Gamma\left(\frac{a_1}{\epsilon_1}\right)}{\Gamma\left(\frac{a_1 - z}{\epsilon_1}\right)} \frac{{}_0F_1\left(\frac{a_1 - z}{\epsilon_1}, -\frac{q}{{\epsilon_1}^2}\right)}{{}_0F_1\left(\frac{a_1}{\epsilon_1}, -\frac{q}{{\epsilon_1}^2}\right)}$$

It is straightforward to expand $w(z)$ around $q = 0$. The result up to third order is:

$$w(z) = \frac{q}{-a+z+\epsilon_1} - \frac{q^2}{(a-z)\,(-a+z+\epsilon_1)^2} - \frac{2q^3}{(a-z)\,(a-z-\epsilon_1)^3\,(a-z+\epsilon_1)} - \frac{(5a-5z+\epsilon_1)\,q^4}{(a-z)^2\,(a-z+\epsilon_1)} + \cdots$$

At $\epsilon_1 = 0$ this series coincides with the small q expansion of

$$w_0(z) = \frac{z-a-\sqrt{(z-a)^2-4q}}{2}$$

as expected. We see that in the limit $\epsilon_1 \to 0$ the poles of $w(z)$ condense around a giving rise to above brunch cut. This is a generic phenomenon, in the case of the gauge group $U(N)$ in small ϵ_1 limit the familiar N brunch cuts around the expectation values would emerge.

Conclusions

To summarise, a saddle point analysis of the instanton series for the Nekrasov partition function in the limit $\epsilon_2 \to 0$ is performed. The criticality condition can be consistently truncated to a finite system of Bethe-Ansatz type equations considering array of Young tableau with number of columns less or equal to L. The truncated system with fixed L determines all quantities up to the instanton order q^L. In large L limit this system of algebraic equations is equivalent to a functional equation for an entire function whose zeros carry information about the lengths of columns of the Young tableau. This functional equation resembles Baxter's equation for 2 d integrable systems which also emerges in the context of 2d integrable field theories Bazhanov,Lukyanov,Zamolodchikov 'arXiv:hep-th/9412229 .

After a simple transformation it becomes evident that this functional equation represents a direct generalization of the algebraic equation defining the Seiberg-Witten curve. The analogue of the SW differential and its relation to the prepotential and chiral correlation functions is established. In particular it is shown that the derivative of the (ϵ_1 deformed) prepotential with respect to the gauge coupling is simply the sum of all column lengths.

Finally, the simplest $U(1)$ case is solved analytically making use of the Gauss' method of the continued fractions.

It would be interesting to find Thermodynamic Bethe-Ansatz (TBA) like equations corresponding to our functional relation thus establishing direct contact with the results of the paper

Nekrasov,Shatashvili 'arXiv:0908.4052.

THANKS