Reflections on Fuzzy Spaces, Landau Levels & Gravity

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Fuzzy spaces provide approximations to a differential manifold in terms of finite-dimensional matrices. What can they tell us about gravity?

- Fuzzy spaces-quantum Hall effect connection
- Extending the quantum Hall effect to higher dimensions, primarily to $\mathbb{CP}^k$
  - Lowest Landau level as a fuzzy space, copy of $\mathbb{CP}^k$
  - Dynamics for the lowest Landau level
    - Bulk dynamics $\Longrightarrow$ Kähler-Chern-Simons action
    - Edge dynamics $\Longrightarrow$ Generalized WZW action
- General result for the large N limits of the Chern-Simons one-form

$$\int dt\ \mathrm{Tr}\ D_0 \quad \Longrightarrow \quad S_{CS}(A_0, A_i)$$

Matrix model — Continuous field theory

$A_0,\ A_i$ parametrize the different large N limits

- Comment on relation to Bergman metric
- Gauge fields correspond to gauging of isometries $\Longrightarrow$ gravity
- Evolution of states for space $\sim$ evolution of states for matter
- Fuzzy spaces lead to Chern-Simons gravity (almost unique)
- Gravity arises as an optimization: How do we choose the "best" large N limit to simplify the dynamics of other (matter) systems?
- Comment on how Minkowski signature can arise

- Fuzzy spaces can be defined by the triple $(\mathcal{H}_N, Mat_N, \Delta_N)$
    - $\mathcal{H}_N = N$-dimensional Hilbert space
    - $Mat_N =$ matrix algebra of $N \times N$ matrices which act as linear transformations on $\mathcal{H}_N$
    - $\Delta_N =$ matrix analog of the Laplacian.
- In the large N approximation
    - $\mathcal{H}_N \longrightarrow$ Phase space $\mathcal{M}$
    - $Mat_N \longrightarrow$ Algebra of functions on $\mathcal{M}$
    - $\Delta_N \longrightarrow$ needed to define metrical and geometrical properties.
- $\mathcal{M}_F \equiv (\mathcal{H}_N, Mat_N, \Delta_N)$ defines a noncommutative and finite mode approximation to $\mathcal{M}$.
- Quantum Hall Effect on a compact space $\mathcal{M}$, lowest Landau level $\sim \mathcal{H}_N$
- Observables restricted to the lowest Landau level $\in Mat_N$
- Can we utilize this to study fuzzy spaces by analyzing QHE?

- Consider the $(n+1)\times(n+1)$ angular momentum matrices J^a, $n = 2j$
- Define

$$X^a = \frac{J^a}{\sqrt{j(j+1)}}$$

- These obey

$$X^a X^a = 1$$

- Functions of these matrices are functions of $\mathbf{1},\ X^a,\ X^{(a}X^{b)} - \frac{1}{3}\delta^{ab}, \cdots$; there are $(n+1)^2$ independent functions for a basis.
- This agrees with

$$f(S^2) = \sum_0^n f_{lm} Y^l_m(\theta,\varphi), \qquad \sum_0^n (2l+1) = (n+1)^2$$

- Further, when $n \to \infty$,

$$[X^a, X^b] = i\epsilon^{abc}\frac{X^c}{\sqrt{j(j+1)}} \Longrightarrow 0$$

- HU AND ZHANG introduced QHE on S^4 where the background magnetic field = $SU(2)$ "instanton"
- We will start by generalizing to arbitrary even dimensions
- QHE on $\mathbb{CP}^k$ ($U(1)$ and $SU(k)$ background fields) (KARABALI, NAIR)
- $\mathbb{CP}^k$ is given as

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)} \sim \frac{SU(k+1)}{U(1) \times SU(k)}$$

- This allows the introduction of constant background fields which are valued in $\underline{U(k)} \sim \underline{U(1)} \oplus \underline{SU(k)}$
- Useful comparison:

$$\text{Minkowski} = \text{Poincaré}/\text{Lorentz}$$

- Since $\mathbb{CP}^1 \sim S^2 = SU(2)/U(1)$, start with choosing $g = \exp(i\sigma \cdot \theta/2) \in SU(2)$ as coordinates for the space (and a gauge direction).
- Wave functions are given by the Wigner $\mathcal{D}$-functions

$$\mathcal{D}^{(j)}_{ms}(g) = \langle j, m | \exp(iJ \cdot \theta) | j, s \rangle$$

  subject to a condition on s.
- Define right translations as $R_a\, g = g\, t_a$.
- The covariant derivatives $D_\pm = iR_\pm/r$. Since

$$[R_+, R_-] = 2R_3 \qquad \Longrightarrow \qquad [D_+, D_-] = -\frac{2R_3}{r^2}$$

  we must choose R_3 to be $-n$ for the Landau problem.
- This corresponds to a field $a = in\,\mathrm{Tr}(t_3 g^{-1} dg)$.

- The wave functions are thus

$$\Psi_m(g) \sim \mathcal{D}^{(j)}_{m,-n}(g)$$

- Choose the Hamiltonian as

$$\mathcal{H} = \frac{1}{4mr^2}\left[R_+ R_- + R_- R_+\right]$$

- The left action

$$L_a\, g = t_a\, g$$

  commutes with $\mathcal{H}$ and corresponds to "magnetic translations".

- The lowest Landau level (LLL) has the further condition (holomorphicity condition)

$$R_- \Psi_m(g) = 0$$

- LLL states also correspond to co-adjoint orbit quantization of $a = in\, \mathrm{Tr}(t_3 g^{-1} dg)$.

- On $\mathbb{CP}^k$ one can have "constant" background magnetic fields in $U(1)$ or $U(k)$ (field strengths $\sim$ Riemannian curvature $\sim$ $U(k)$ structure constants)

$$\mathbb{CP}^k = SU(k+1)/U(k) \sim SU(k+1)/\{U(1) \times SU(k)\}$$

- Parametrize using the $(k+1) \times (k+1)$ matrix, $g \in SU(k+1)$, with $g \sim gh$, $h \in U(k)$
- The constant fields correspond to

$$a = in\sqrt{\frac{2k}{k+1}}\, \mathrm{Tr}(t_{k^2+2k}\, g^{-1} dg), \qquad \text{U(1) field}$$

$$\bar{A}^a = 2i\, \mathrm{tr}(t^a g^{-1} dg), \qquad \text{SU(k) field}$$

t_A →

$$t_a \quad \subset SU(k), \quad a = 1, \cdots, k^2 - 1$$

$$t_{k^2+2k} \quad \subset U(1)$$

$$t_\alpha \quad \subset \text{coset} \qquad t_{+I},\ t_{-I}$$

- Wave functions form $SU(k+1)$ representations; expressed in terms of Wigner $\mathcal{D}$-functions

$$\Psi \sim \mathcal{D}^{(J)}_{L,R}(g) = \langle L \mid \hat{g} \mid R \rangle$$

quantum numbers characterizing states in J-representation

- Abelian case ($U(1)$ background field)

Under $U(1)_R$: $\quad a \to a - \frac{nk}{\sqrt{2k(k+1)}} d\theta$

Under $SU(k)_R$: $\quad a \to a$

$$\Psi_m \sim \langle \, m \mid \hat{g} \mid \underbrace{R_a = 0, R_{k^2+2k} = -\frac{nk}{\sqrt{2k(k+1)}}}\, \rangle$$

$m = 1, \cdots, \dim J$

$SU(k)_R$ singlet with fixed $U(1)_R$ charge

- Nonabelian case ($U(k)$ background field)

  $\bar{A}^a$ transforms under $SU(k)_R \to$ wave functions carry $SU(k)_R$ charge

$$\Psi_m^{a'} \sim \langle\, m \,|\, \hat{g} \,| R \,\rangle$$

$m = 1, \cdots, \dim J$

$SU(k)_R$ repr. J' with fixed $U(1)_R$ charge

a' internal gauge index $=1, \cdots, N' = \dim J'$

- The Hamiltonian can be taken as

$$\begin{aligned} H &= \frac{1}{2MR^2} \sum_{I=1}^{k} R_{+I} R_{-I} + \text{constant} \\ &= \frac{1}{2MR^2} \left[C_2^{SU(k+1)}(J) - C_2^{SU(k)}(J') - \frac{n^2 k}{2(k+1)} \right] \end{aligned}$$

- For the lowest Landau level, $R_{-I}\Psi = 0$ (holomorphicity condition).

- In the Abelian case

$$\Psi \in \text{symmetric rank } n \text{ representation } J$$
$$N = \dim J = \frac{(n+k)!}{n!k!} \rightarrow \frac{n^k}{k!}$$

- These are coherent states for $\mathbb{CP}^k$
- Think of $\mathbb{CP}^k$ as a phase space, quantization leads to the finite dimensional Hilbert space of LLL states.
- LLL of $\mathbb{CP}^k$ with $U(1) \equiv$ "fuzzy" $\mathbb{CP}^k$
- In the large N limit, matrices which are operators on LLL states become functions on $\mathbb{CP}^k$
- This gives an approach to building smooth spaces as large N limits of finite-dimensional Hilbert spaces
- A similar story for the nonabelian case.

Matrix formulation of LLL dynamics

- The LLL has N available states, K occupied by fermions, $1 \ll K \ll N$
- Form a QH droplet, specified by the density matrix: $\hat{\rho}_0 = \sum_{i=1}^{K} |i\rangle\langle i|$,

$$\hat{\rho}_0 = \left[\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & 0 & \\ & & & & 0 \end{array} \right] \begin{array}{l} \updownarrow K \\ \updownarrow N-K \end{array}$$

- Under time evolution: $\hat{\rho}_0 \rightarrow \hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^\dagger$

  $\hat{U} = N \times N$ unitary matrix: "collective" variable describing all LLL excitations

- The dynamics is given by

$$S = \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \, - \, \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right]$$

- This leads to the evolution equation for density matrix

$$i\frac{d\hat{\rho}}{dt} = [\hat{V}, \hat{\rho}]$$

(No explicit dependence on properties of space on which QHE is defined, abelian or nonabelian nature of fermions etc)

- The symbol for a matrix is

$$\begin{aligned} X(\vec{x}, t) &= \frac{1}{N}\sum_{m,l} \Psi_m(\vec{x}) X_{ml}(t) \Psi^*_l(\vec{x}) \\ X^{a'b'}(\vec{x}, t) &= \frac{1}{N}\sum_{m,l} \Psi^{a'}_m(\vec{x}) X_{ml}(t) \Psi^{*b'}_l(\vec{x}), \qquad a', b' = 1, \cdots N' = \dim J' \end{aligned}$$

- We seek a simplification at large N in terms of the symbol for U.

- This utilizes the usual rules,

$$\begin{aligned}
\underbrace{\hat{\rho}_0,\, \hat{U},\, \hat{V}}_{(N\,\times\,N)\text{ matrices}} &\quad\Longrightarrow\quad \underbrace{\rho_0(\vec{x}),\, U(\vec{x},t),\, V(\vec{x})}_{\text{Symbols}}\\
\text{Matrix multiplication} &\quad\Longrightarrow\quad *\ \text{product}\\
\text{Trace operation Tr} &\quad\Longrightarrow\quad N\int d\mu\\
\left(\hat{O}_1\,\hat{O}_2\right)_{symbol} &\quad=\quad O_1(\vec{x},t) * O_2(\vec{x},t)
\end{aligned}$$

- For example, the star product can be written as

$$\begin{aligned}
A(g) * B(g) \quad=\quad & \sum_s (-1)^s \left[\frac{(n-s)!}{n!s!}\right] \sum_{i_1+i_2+\cdots+i_k=s}^{n} \frac{s!}{i_1!i_2!\cdots i_k!}\, \hat{R}^{i_1}_{-1}\hat{R}^{i_2}_{-2}\cdots\hat{R}^{i_k}_{-k}A(g)\\
& \times \hat{R}^{i_1}_{+1}\hat{R}^{i_2}_{+2}\cdots\hat{R}^{i_k}_{+k}B(g)
\end{aligned}$$

- Bosonic action can be written in terms of $G \in U(N')$

$$S = \frac{1}{4\pi}\int_{\partial D} \mathrm{Tr}\left[\left(G^\dagger \dot{G} + \omega\, G^\dagger \mathcal{L} G\right) G^\dagger \mathcal{L} G\right]$$
$$+\frac{1}{4\pi}\int_D \mathrm{Tr}\Big[-d\left(i\bar{A} dG G^\dagger + i\bar{A} G^\dagger dG\right) + \underbrace{\frac{1}{3}\left(G^\dagger dG\right)^3\Big]\left(\frac{\Omega}{2\pi}\right)^{k-1}}_{\text{WZW-term in } 2k+1 \text{ dim}}$$

$\mathcal{L} = \frac{1}{n}(\Omega^{-1})^{ij}\hat{r}_j \mathcal{D}_i \phi$ = covariant derivative along the boundary droplet

- In the presence of gauge interactions

$$\begin{aligned} S &= N\int dt\, d\mu\, \mathrm{tr}\left[i\rho_0 * U^\dagger * \partial_t U - \rho_0 * U^\dagger * (V + \mathcal{A}) * U\right] \\ &= S_{\text{edge}} + S_{\text{bulk}} \end{aligned}$$

- Invariance under $U(N)$ rotations $\delta \hat{U} = -i\hat{\lambda}\, \hat{U}$ implies that S is invariant under

$$\delta U = -i\lambda * U$$
$$\delta \mathcal{A}(\vec{x}, t) = \partial_t \lambda(\vec{x}, t) - i\left(\lambda * (V + \mathcal{A}) - (V + \mathcal{A}) * \lambda\right)$$

- We need the above transformation to be induced by

$$\begin{aligned} \delta A &= \partial\Lambda + i[\bar{A} + A, \Lambda] \\ \mathcal{A} &= \text{function}(A_\mu,\, \bar{A}_\mu,\, V) \\ \lambda &= \text{function}(\Lambda, A_\mu, \bar{A}_\mu) \end{aligned}$$

$$\begin{aligned} \mathcal{A} = A_0 &- \frac{i}{2n} g^{ij}\,[A_i,\, 2D_iA_0 - \partial_0 A_i + i[A_i,\, A_0]] + \frac{1}{4n}(\Omega^{-1})^{ij}\{A_i, 2D_jA_0 - \partial_0 A_j + i[A_j,\, A_0]\} \\ &+ u^i A_i - \frac{i}{2n} g^{ij}\,[A_i,\, A_k]\,\partial_j u^k + \frac{1}{4n}(\Omega^{-1})^{ij}\{A_i,\, A_k\}\partial_j u^k \\ &- \frac{i}{2n} g^{ij}\,\left[A_i,\, 2D_jA_k - D_kA_j + i[A_j,\, A_k]\; + 2\bar{F}_{jk}\,\right] u^k \\ &+ \frac{1}{4n}(\Omega^{-1})^{ij}\{A_i,\, 2D_jA_k - D_kA_j + i[A_j,\, A_k] + 2\bar{F}_{jk}\,\}u^k \\ &+ \frac{1}{2n^2} g^{ik}(\Omega^{-1})^{jl}\left(\mathcal{D}_iA_j + \mathcal{D}_jA_i\right)\nabla_k\partial_l V + \cdots \end{aligned}$$

where $u^i = \frac{1}{n}(\Omega^{-1})^{ij}\partial_j V$

- Relation between $\mathcal{A}$ and A is essentially the Seiberg-Witten transformation

- $S_{\rm edge} \sim S_{\rm WZW}\left(A^L = A + \bar{A}\,, A^R = \bar{A}\right)$

  = chirally gauged WZW action generalized in $2k$ $\Big(\partial$(droplet) + time$\Big)$ dimensions

- The bulk action is

$$S_{\rm bulk} = \frac{(-1)^{k+1}}{(2\pi)^k k!} \int {\rm tr}\Bigg[A\,(-n\Omega)^k + \frac{k}{2}\left((A+\bar{A}+V)d(A+\bar{A}+V) + \frac{2i}{3}(A+\bar{A}+V)^3\right)(-n\Omega)^{k-1} + \frac{k(k-1)}{2}\left((A+\bar{A})d(A+\bar{A}) + \frac{2i}{3}(A+\bar{A})^3\right)dV\,(-n\Omega)^{k-2}\Bigg] + \cdots$$

(KARABALI; both $S_{\rm edge}$ and $S_{\rm bulk}$ related to the KCS actions of NAIR, SCHIFF)

- The bulk action is a CS action, $S_{\rm bulk} \sim S^{2k+1}_{CS}(\tilde{A})$

  $\tilde{A} = (A_0 + V, a_i + \bar{A}_i + A_i)$ = background + fluctuations

- Gauge Invariance ⇒ Anomaly Cancellation

$$\delta S_{\text{edge}} \neq 0, \quad \delta S_{\text{bulk}} \neq 0$$

$$\delta S_{\text{edge}} + \delta S_{\text{bulk}} = 0$$

- The edge action for S^4 case obtained by using the fact that $\mathbb{CP}^3$ is locally $S^4 \times \mathbb{CP}^1$.
- The excitations do not have Lorentz invariance

The bulk fields are gauging the isometries of the space; hence they should be interpreted in terms of gravity on the fuzzy space.

- A deformation of background is of the form

$$\Omega \implies \Omega + F$$

- Classically, we can ask: Dynamics given by $(\mathcal{H}, \Omega + F)$ equivalent to $(\tilde{\mathcal{H}}, \Omega)$?
- In quantum theory, the Hilbert space for $\Omega + F$ is the same if the characteristic class of Ω is unchanged; for example, in two dimensions if

$$\int \Omega + F = \int \Omega$$

- However, the wave functions can be modified. This leads to new symbols

$$X(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}, A) X_{ml}(t) \Psi_l^*(\vec{x}, A)$$

- Introduction of background fields leads to new wave functions, new symbols, new large N limit or, turning this around, large N limits can be labeled by possible background fields.

- The change due to change in A can be obtained in two ways:
  - Work out changes in $\Psi_m(\vec{x}, A)$ and the corresponding changes in the symbol

OR

  - We can write a general matrix function as sums of monomials of the form

$$K = K^{\mu_1 \mu_2 \cdots \mu_n} D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n}$$

and work out changes as we shift $D \to D + \delta A$

($K = D_0$ wil be needed for the effective action.)

- For a shift of D_μ we can write

$$\delta D_\mu = \frac{1}{2}\Big[\xi^\alpha [D_\alpha, D_\mu] + [D_\alpha, D_\mu] \tilde{\xi}^\alpha \Big]$$

$$\xi^\alpha = \delta D_\lambda (\Omega^{-1})^{\lambda\alpha}, \quad \tilde{\xi}^\alpha = (\Omega^{-1})^{\lambda\alpha} \delta D_\lambda, \quad \Omega_{\mu\nu} = [D_\mu, D_\nu]$$

- For the change of K under a shift of D_μ, we get

$$\begin{aligned}
\delta K &= \tfrac{1}{2}\left[\delta_1 K + \delta_2 K\right] \\
\delta_1 K &= \xi^\alpha [D_\alpha, K] + \sum_{k=1}^{n-1} D_{\mu_1} \ldots D_{\mu_{k-1}} [D_{\mu_k}, \xi^\alpha][D_\alpha, K^{\mu_1 \cdots \mu_k}] \\
\delta_2 K &= [K, D_\alpha]\tilde{\xi}^\alpha + \sum_{n}^{2} [\tilde{K}^{\mu_k \cdots \mu_n}, D_\alpha][\tilde{\xi}^\alpha, D_{\mu_k}] D_{\mu_{k+1} \ldots D_{\mu_n}}
\end{aligned}$$

- The $K^{\mu_1 \cdots \mu_k}$ are determined iteratively by recursion rules

$$\begin{aligned}
K^\mu &= (\Omega^{-1})^{\mu\lambda}[D_\lambda, K] - (\Omega^{-1})^{\mu\lambda} D_\nu [D_\lambda, K^\nu] \\
K^{\mu\nu} &= (\Omega^{-1})^{\nu\lambda}[D_\lambda, K^\mu] - (\Omega^{-1})^{\nu\lambda} D_\alpha [D_\lambda, K^{\mu\alpha}] \\
\cdots & \quad \cdots \quad \cdots \quad \cdots
\end{aligned}$$

And similarly for $\tilde{K}^{\mu_k \cdots \mu_n}$.

- The bulk action is given by

$$\begin{aligned} S &= i\mathrm{Tr}\left(\hat{\rho}_0 \hat{U}^\dagger D_0 \hat{U}\right) \\ &= i\mathrm{Tr}\left(\hat{\rho}_0 \hat{U}^\dagger \partial_0 \hat{U}\right) - \mathrm{Tr}\left(\hat{\rho}_0 \hat{U}^\dagger \hat{A}_0 \hat{U}\right) \end{aligned}$$

  where we can take $\hat{\rho}_0 = \mathbf{1}$.

- For example, for $\mathbb{CP}^1$, the variation is given by

$$\delta S = i\mathrm{Tr}(\hat{\rho}_0 \delta D_0) \approx i\mathrm{Tr}[\delta D_\mu (\Omega^{-1})^{\mu\nu} F_{\nu 0}]$$

  Integration of this will give the action.

- We take the large n limit, taking a background $U(1)$ field (corresponding to the symplectic form) and fluctuations which may be nonabelian. i.e.,

$$\Omega^{-1} \approx \omega^{-1} - \omega^{-1} F \omega^{-1} + \cdots$$

  F is the fluctuation from the background value ω.

- There is also a change in the symbol of a product,

$$(AB)_0 = (AB) - \tfrac{1}{2}\,\mathrm{Tr}\,[\omega^{-1})^{\mu\nu} F_{\mu\nu}(AB + BA)]$$

- The effective action becomes, say, for $\mathbb{CP}^2$,

$$\begin{aligned}\delta S \;&=\; \int \det\omega \left[\tfrac{1}{2}\,\mathrm{Tr}\,[\delta A_\mu F_{\nu 0} + F_{\nu 0}\delta A_\mu)](\omega^{-1})^{\mu\nu} \right.\\ &\quad - \tfrac{1}{4}\,\mathrm{Tr}\,[\delta A_\alpha (F_{\beta 0} F_{\mu\nu} + F_{\mu\nu} F_{\beta 0})](\omega^{-1})^{\alpha\beta}(\omega^{-1})^{\mu\nu}\\ &\quad \left. - \tfrac{1}{2}\,\mathrm{Tr}\,[\delta A_\alpha (F_{\beta 0} F_{\mu\nu} + F_{\mu\nu} F_{\beta 0})][(\omega^{-1})^{\alpha\mu}(\omega^{-1})^{\nu\beta} \right]\end{aligned}$$

- Integration of this leads to the action

$$\begin{aligned} S \;&=\; \int \left[\omega\wedge\omega\wedge A + \omega\wedge (C.S.)^{(3)} + \frac{1}{3}(C.S.)^{(5)} \right]\\ &=\; \mathcal{S}_{CS}(\mathcal{A}), \qquad\qquad \mathcal{A} = a + A, \qquad da = \omega \end{aligned}$$

- The general result is

$$
\begin{aligned}
i \int dt\, \mathrm{Tr}(D_0) &\approx S_{*CS}^{(2k+1)}(a+A) + \cdots, \quad \text{as } N \to \infty \\
&\approx S_{CS}^{(2k+1)}(a+A)
\end{aligned}
$$

(For the Abelian case, this is related to Moser's lemma.)

- The latter form is background independent, just like the matrix action $i\mathrm{Tr}\left(\hat{\rho}_0 \hat{U}^\dagger D_0 \hat{U}\right)$. The expansion of the matrix action in terms of different backgrounds is obtained, in the large n limit, by expanding the CS action around the corresponding gauge potentials.
- This is a general matrix result, the CS one-form can generate all the higher CS forms as appropriate large N limits
- Before we turn to gravity, we comment on how this is related to the Bergman metric

- The density ρ can be written in terms of the wave functions as

$$\rho = \frac{1}{N} \sum_m \Psi_m(\vec{x}, A) \Psi_m^*(\vec{x}, A)$$

- The Bergman metric for Kähler manifolds is given by

$$g = \frac{1}{n} \partial \bar{\partial} \log \rho$$

  The expansion of this in powers of curvatures is important for approximating Einstein metrics for Kähler manifolds, for example, for Calabi-Yau manifolds in $\mathbb{CP}^k$.

- TIAN, YAU &ZELDITCH and LU & CATLIN derived the expansion

$$\rho \approx \omega^k + \omega^{k-1} \frac{R}{2} + \omega^{k-1} \left(\frac{1}{3} \Delta R + \frac{1}{24} |Riem|^2 - \frac{1}{6} |Ric|^2 + \frac{1}{8} R^2 \right) + \cdots$$

- More recently, DAI, LIU & MA and MA & MARINESCU obtained

$$\rho \approx \omega^k + \omega^{k-1} \left(\frac{R}{2} \mathbf{1}_E + i R_E \right) + \cdots$$

- These results (and some higher terms) are reproduced by our results by taking

$$\rho = \frac{\delta S}{\delta A_0}$$

and expanding around ω.

Can we systematize this?

- The gauge fields in

$$i \int dt\, \mathrm{Tr}(D_0) \approx S_{CS}^{(2k+1)}(a + A) + \cdots\,, \quad \text{as } N \to \infty$$

lead to gauging of the isometry group $SU(k+1)$ of $\mathbb{CP}^k$, so a natural interpretation is in terms of gravity

> We will take an approach of starting with the Hilbert space of (Matter +Gravity), an already quantized theory, and extracting the notion of continuous spacetime in the large N limit.

- Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_m$, general state

$$|\, A,\; r\, \rangle$$

Correspond to "space" ← A ; r → Correspond to "matter"

- For D_0, make an ansatz

$$\langle A, r|D_0|B, s\rangle = \delta_{rs}\, \langle A|D_0^{(s)}|B\rangle \;+\; \langle A, r|D_0^{(m)}|B, s\rangle$$

$$\langle A, r|\rho_0|B, s\rangle = \delta_{AB}\, \langle r|\rho_0|s\rangle$$

- A_0 (or H) specifies the choice of matter system. For spacetime, the geometry is not a priori determined
- $D^{(s)}$ should be regarded as an arbitrary matrix
- Entropy of de Sitter space, $e^S \sim \exp(1/\Lambda) \Rightarrow$ There are states in the Hilbert space representing pure space
- Dynamics of space should be treated exactly as dynamics of matter
- Action, as for any quantum theory, is given by

$$S = i \int dt\, \mathrm{Tr}(\rho\, U^\dagger D_0 U)$$

- Optimize the choice of large N limit $\Longrightarrow$ Extremization of the action (with respect to $D_0^{(s)}$) to determine the "best" background.
- If we ignore all matter degrees of freedom as a first approximation, the action becomes

$$S \approx i \int dt \, \mathrm{Tr}(D_0^{(s)})$$

- For large number of states, the action is effectively S_{CS}. Thus,

Fuzzy spaces (Matrix models, QH model) $\Longrightarrow$ Chern-Simons gravity

- Indications of CS gravity action in M-theory
- As an example, take a large N limit which leads to the 7-dim. CS action, starting with $\mathbb{CP}^3 \times \mathbb{R}$
- Gauge group $\sim U(4) \sim SO(6) \times U(1)$

- Choose $\mathcal{M}^7 = \mathcal{M}^5 \times S^2$, with $F_{U(1)} = l\,\omega$, where ω is the Kähler form on S^2.
- The effective large N action is

$$
\begin{aligned}
S &= -i\frac{l}{24\pi^2}\int \mathrm{Tr}\left(A\,dA\,dA + \frac{3}{2}A^3\,dA + \frac{3}{5}A^5\right)\\
A &= \mathcal{P}^a\, e^a_\mu\, dx^\mu \;+\; \tfrac{1}{2}\, J^{ab}\, \omega^{ab}_\mu\, dx^\mu
\end{aligned}
$$

$O(6)/O(5)$ $O(5)$

- Euclidean de Sitter space is a solution
- A further choice $e^5_5 = 1$, $\omega^{5a} = 0$, $\omega^{ab}_5 = 0$, for $a, b = 1, \ldots, 4$, leads to the Einstein action in 4 dimensions,

$$
S = \frac{l\Lambda}{16\pi}\int \sqrt{g}\; d^4x\,(R - 3\Lambda)
$$

- This is similar to the McDowell-Mansouri formulation of Einstein gravity.
- Key points to be clarified:
  - How does Minkowski signature arise?
  - How is (quantized) matter coupled to gravity?

  There are partial answers to both, but details are yet to be worked out.
- It is not clear if we will have holography.
- There is no issue of quantizing a classical theory of gravity, we start with the Hilbert space.
- Spacetime is nothing more than a convenient framework for formulating matter interactions.