INTERNAL STRUCTURE OF STRING VACUA:

from **NON-**geometric susy to **NON-**supersymmetric geometry

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SUPERSYMMETRIC COMPACTIFICATIONS WITHOUT FLUXES

10D string theory on $ds_{10}^2 = \mathrm{d}s_4^2(M_4) + \mathrm{d}s_6^2(X)$

- SUSY $\Rightarrow$ covariantly constant spinor
- $\Leftrightarrow$ $\left\{ \right.$
  - ◇ real two-form J
  - ◇ complex three-form Ω ($c_1(X) = 0$)
  - ◇ $J \wedge \Omega = 0$ and $\mathrm{d}J = \mathrm{d}\Omega = 0$
- X - is a manifold of $SU(3)$ holonomy - CY (Kähler with $c_1(X) = 0$)

D-branes

(calibrated submanifolds with stable bundles) :

- ◇ B: $\mathrm{Im}(e^{i\alpha} e^{-iJ+\mathcal{F}}) = 0$
- ◇ A: $\mathrm{Im}(e^{i\alpha} \Omega \wedge \mathcal{F}^l) = 0$
- ◇ mirror symmetry: A $\Leftrightarrow$ B

SUPERSYMMETRIC FLUX BACKGROUNDS

10D string theory with

- the fluxes :
  - NS 3-form : $\mathrm{d}H = 0$
  - RR (even/odd - IIA/B) : $F^{(10)} = F + \mathrm{vol}_4 \wedge \lambda(*F)$ $\quad (\lambda(F_n) = (-1)^{\mathrm{Int}[n/2]} F_n)$
- the metric: $ds_{10}^2 = e^{2A(y)} \mathrm{d}s_4^2(M_4) + \mathrm{d}s_6^2$

Equations of motion $\Leftrightarrow$

- susy $\Leftrightarrow$ pure spinor equations

$$\mathrm{d}\left(e^{3A}\Phi_1\right) = 0 \qquad \Leftrightarrow \qquad \text{Gen. CY structure}$$

$$\mathrm{d}\left(e^{2A}\mathrm{Re}\Phi_2\right) = 0$$

$$\mathrm{d}\left(e^{4A}\mathrm{Im}\Phi_2\right) = e^{4A} e^{-B} * \lambda(F)$$

$$\langle \Phi_1, \gamma \Phi_2 \rangle = 0$$

- Bianchi identities

$$(\mathrm{d} - H\wedge)F = \delta(\mathrm{source})$$

Φ_1 and Φ_2 are even/odd poly-forms for IIA/B: IIA $\rightarrow$ $\begin{array}{c}\Phi_1 = \Phi_+ \\ \Phi_2 = \Phi_-\end{array}$ IIB $\rightarrow$ $\begin{array}{c}\Phi_1 = \Phi_- \\ \Phi_2 = \Phi_+\end{array}$

What are $\Phi_\pm$?

$$\Phi_+ = 8\, e^{-\phi} e^{-B} |\eta_+^1 \otimes \eta_+^{2\,\dagger}|_{\text{norm}} \longmapsto e^{i\theta_+} e^{-\phi} e^{-B}\, e^{-iJ}$$

$$\Phi_- = 8\, e^{-\phi} e^{-B} |\eta_+^1 \otimes \eta_-^{2\,\dagger}|_{\text{norm}} \longmapsto -ie^{i\theta_-} e^{-\phi} e^{-B} \Omega$$

($SU(3)$ structure manifolds)

$$\langle \Phi_1, \gamma \Phi_2 \rangle = 0 \longmapsto J \wedge \Omega = 0, \quad \text{and} \quad J^3 = \frac{3i}{4} \Omega \wedge \bar{\Omega}$$

(use: $\eta_+^1 \otimes \eta_\pm^{2\,\dagger} = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta_\pm^{2\,\dagger} \gamma_{m_k \ldots m_1} \eta_+^1 \right) \gamma^{m_1 \ldots m_k}$ for spinors

IIA $\rightarrow$ $\begin{array}{l}\epsilon_1 = \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1 \\ \epsilon_2 = \zeta_+ \otimes \eta_-^2 + \zeta_- \otimes \eta_+^2\end{array}$ IIB $\rightarrow$ $\begin{array}{l}\epsilon_1 = \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1 \\ \epsilon_2 = \zeta_+ \otimes \eta_+^2 + \zeta_- \otimes \eta_-^2\end{array}$)

Generalized complex structure (GCG)

- GCG $\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$ $\quad (\mathcal{J}^2 = -1; \quad \mathcal{J}^\dagger \mathcal{I} \mathcal{J} = \mathcal{I})$
  - ◇ Structure group: $\Rightarrow$ U(3,3)
- GCS integrable: $\pi_+[\pi_-(v), \pi_-(w)]_{\text{Lie}} = 0 \;\mapsto\; \Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0$ with Courant bracket:

$$[v+\xi,\, w+\eta] = [v, w]_{\text{Lie}} + \left\{ \mathcal{L}_v \eta - \mathcal{L}_w \xi - \tfrac{1}{2}\, \mathrm{d}(\imath_v \eta - \imath_w \xi) \right\}$$

(Courant closes on $L_{\mathcal{J}}$ – the i-eigenbundles of $\mathcal{J}$.)

- Closed B-transform $(v_1, \rho_1) \mapsto e^B(v_1, \rho_1) = (v_1, \rho_1 + \imath_{v_1} B)$ is an auto-morphism of Courant : $[e^B(v_1, \rho_1), e^B(v_2, \rho_2)] \mapsto e^B[(v_1, \rho_1), (v_2, \rho_2)]$
- Twisting: $\mathrm{d} \mapsto \mathrm{d} - H\wedge, \qquad [.,.]_C \mapsto [.,.]_C + \underline{\imath_v \imath_w H}$
- Two compatible GCS's (Structure group: $\Rightarrow$ U(3)×U(3)): $[\mathcal{J}_1, \mathcal{J}_2] = 0 \quad \Rightarrow$

$$G = \mathcal{J}_1 \mathcal{J}_2 = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$$

Pure spinors and integrability of GCS (GCY)

- i-eigenbundle of $\mathcal{J}$ $L_{\mathcal{J}}$ is of max dimension (6) and is null - max. isotropic.
- Spinor bundle $S \sim \Lambda^{\bullet} T^{*}$:
  - ◇ Clifford action on a spinor Φ: $(v+\zeta)\cdot\Phi = v^{m}\imath_{\partial_m}\Phi + \zeta_m \mathrm{d}x^m \wedge \Phi$
  - ◇ $L_{\Phi} = \{v+\zeta \in T\oplus T^{*} \,|\, (v+\zeta)\cdot\Phi = 0\}$ is isotropic
    ($((v+\zeta)\cdot(v+\zeta)\Phi) = -(v+\zeta, v+\zeta)\Phi$)
  - ◇ If L_{Φ} of max. dimension – Φ - pure spinor
  - ◇ If $L_{\mathcal{J}} = L_{\Phi} \quad \Rightarrow \quad \mathcal{J} \leftrightarrow$ line of pure spinors
- For $A, B \in L_{\Phi}$, $\quad [A,B]_{\mathrm{C}}\Phi = (AB - BA)\cdot \mathrm{d}\Phi$
  - ◇ $\mathrm{d}\Phi = (\iota_v + \zeta\wedge)\Phi \qquad \Leftrightarrow \qquad \mathcal{J}$ -integrable
  - ◇ $\mathrm{d}\Phi = 0$ ($[A,B]_{\mathrm{C}} \in L_{\Phi} = L_{\mathcal{J}}$) – GCY condition

On generalized tangent bundle :

$$0 \longrightarrow T^*M \longrightarrow E \stackrel{\pi}{\longrightarrow} TM \longrightarrow 0\,,$$

$$\Phi_\pm \in L \otimes \Lambda^{\text{even/odd}} T^*M\,.$$

Dilaton ϕ together with g and B needed to define ismorphism between $S^\pm(E)$ and $\Lambda^{\text{even/odd}} T^*M$

Sections of E:

$$X = \begin{pmatrix} v \\ \xi \end{pmatrix} \qquad \longmapsto \qquad X' = e^{-B}X = \begin{pmatrix} \mathbb{I} & 0 \\ -B & \mathbb{I} \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ \xi - i_v B \end{pmatrix}\,.$$

- B-transformed pure spinor $e^B\,\Phi_\pm$ - global : structure group on $TX \oplus T^*X$ SU(3,3); a pair of compatible pure spinors ($\Phi_+ \,\&\, \Phi_-$) – SU(3)$\times$SU(3)

Back to Pure Spinor equations:

- $\mathcal{N}=1$ SUSY $\quad\Leftrightarrow\quad$ GCY
- RR part is completely fixed by (g, B) (e.g. 32 components of RR flux vs. 42 components of SU(3) intrinsic torsion)
  Pure spinor equations $\Leftrightarrow$ supersymmetry (D=6!)
- $\int \langle (d - H\wedge)F,\ e^{3A}\mathrm{Im}\Phi_2\rangle = \ldots = \int e^{4A}\,\langle F,\ *\lambda(F)\rangle$ has definite sign

  ◇ Projection into singlet $\Leftrightarrow$ $\mathrm{Tr}(\delta\mathcal{L}^{(4)}/\delta g_{\mu\nu})$ $\quad\longleftarrow$ TADPOLE

  ◇ Need for negative charge sources $\quad\longrightarrow$ O-planes

  ◇ Individual terms may correspond both to O-planes and D-branes

  ◇ Sources given by:

$$(d - H\wedge)F_{p-3} \equiv c_i\eta^i = Q_i(\mathrm{source})\ \mathrm{vol}^i$$

- 4+6 split lifts the Self-Duality of RR fields

$$F^{10}_n = (-)^{\mathrm{Int}[n/2]} *_{10} F^{10}_{10-n} \Rightarrow F + \mathrm{vol}_4 \wedge \lambda(*F)$$

  (But Pure Spinors are SD: $\quad \not{\Phi}_\pm\,\gamma = i\ \cancel{\lambda(*\Phi_\pm)})$

ORIENTIFOLDS

Orientifold action:

- O3/O7 and O6: $\Omega_{\rm WS}(-)^{F_L}\sigma$
- O5/O9 and O4/O8 $\Omega_{\rm WS}\sigma$
- IIA: $\sigma I = -I$, IIB: $\sigma I = I$

For SU(3) structure:

IIA: $\sigma\Omega_3 = \mp\bar{\Omega}_3 \quad \sigma e^{-iJ} = e^{iJ}$,

IIB: $\sigma\Omega_3 = \mp\Omega_3 \quad \sigma e^{-iJ} = e^{-iJ}$.

In general:

O3/O7	O5	06
$\sigma(\Phi_+) = -\lambda(\bar{\Phi}_+)$	$\sigma(\Phi_+) = \lambda(\bar{\Phi}_+)$	$\sigma(\Phi_+) = -\lambda(\Phi_+)$
$\sigma(\Phi_-) = \lambda(\Phi_-)$	$\sigma(\Phi_-) = -\lambda(\Phi_-)$	$\sigma(\Phi_-) = \lambda(\bar{\Phi}_-)$

The phases in the pure spinors get fixed. For type IIB the compact backgrounds are:

- ◇ Type B: $\pi/2$ (mod π)
- ◇ Type C: 0 (mod π)

3-STEP CONSTRUCTION OF SUPERSYMMETRIC BACKGROUNDS

	NON-COMPACT	COMPACT
STEP 1 (Twisted) GCY structure	$d_H \Phi_1 = 0$	Φ_1 is compatible with involution σ $d_H \Phi_1 = 0$
STEP 2 Metric $SU(3,3) \to S(3) \times SU(3)$	Φ_2 compatible with Φ_1 $d_H \Phi_2 = *F^{\rm RR}$	Φ_2 compatible with Φ_1 and σ $d_H \Phi_2 = *F^{\rm RR}$
STEP 3 Tadpole	compute $F^{\rm RR}$ $d_H F^{\rm RR} = 0$	compute $F^{\rm RR}$ $d_H F^{\rm RR} = sources$

EXAMPLE: CONFORMAL CY BACKGROUND

- X_6 - conformally CY
- $F_3 + \tau H_3$ - (2,1) and primitive — type B
- $F_5 \sim *_6 dA$ (A - warp factor)
- The tadpole: $dF_5 = F_3 \wedge H_3 + sources \neq 0$
  - ◇ Need O3's and D3's
  - ◇ The tadpole is a top-form (singlet)!
- X_6 can be chosen to be (conformally) T^6
- Can use the isometries of the background to perform T-dualities:
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  Ex: Two T-dualities with $F_3, F_5 \to \tilde{F}_3$ (O3,D3 → O5,D5) and a new manifold:

$$\begin{array}{ccc} T^2 & \hookrightarrow & M_6 \\ & & \downarrow \\ & & T^4 \end{array}$$

Nilmanifold - "Twisted torus" — type C

Negative curvature - Not CY

DELOCALIZED SOURCE SOLUTIONS

- The nilmanifold (0,0,0,12,23,14-35):

$$\begin{array}{ccc} S^1_{\{6\}} & \hookrightarrow & M_6 \\ & & \downarrow \\ T^2_{\{4,5\}} & \hookrightarrow & M_5 \\ & & \downarrow \\ & & T^3_{\{1,2,3\}} \end{array}$$

- The metric is given by: ($B = 0$)

$$\begin{aligned} \Omega_3 &= (e^1 - ie^3) \wedge (e^2 + i\tau e^6) \wedge (e^4 + ie^5) \\ J &= -t_1 e^1 \wedge e^3 + t_2 \tau_r e^2 \wedge e^6 + t_3 e^4 \wedge e^5 \end{aligned}$$

- RR 3-form: $F_3 = -(\tau_i e^2 - |\tau|^2 e^6) \wedge \left(t_2 (e^1 \wedge e^4 - e^3 \wedge e^5) + \frac{t_3}{\tau_r} (e^1 \wedge e^5 + e^3 \wedge e^4) \right)$

- Moduli: $\tau = \tau_r + i\tau_i$ - CS, t_1, t_2, t_3 - "Kähler" (get fixed by BI)
  $t_i > 0$ and $1 + |\tau|^2 \geq 2|\tau_r|$ - for positive definite metric

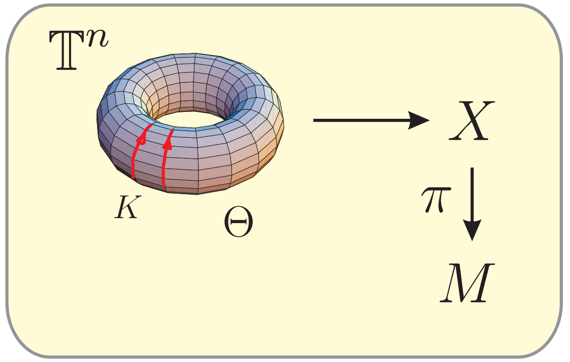
- The tadpole: $dF_3 = -2\,|\tau|^2 \left(\frac{t_3}{\tau_r^2 t_1 t_2} \ \mathrm{vol1}^{1236} + \frac{t_2}{t_1 t_3} \ \mathrm{vol2}^{1345} \right)$
  O5-planes along 45 26 - intersecting sources

- Not T-dual, but ... can be connected to other solutions!

COORDINATE-DEPENDENT O(n,n) TRANSFORMATIONS

$\mathbb{T}^n$ action

- Principal torus bundle $\mathbb{T}^n \hookrightarrow X \xrightarrow{\pi} M$:



- A globally well-defined smooth 1-form Θ on X with values in $\mathfrak{t} := \mathrm{Lie}\, \mathbb{T}^n \cong \mathbb{R}^n$.
- Isometries: $\imath_K \Theta = \mathbb{I} \in \mathfrak{t}^* \otimes \mathfrak{t}$ ($\mathcal{L}_K \Theta = 0$) and $\mathcal{L}_K H = 0$
- 3-form H:

$$H = \pi^* H_3 + \langle \pi^* H_2, \Theta \rangle + \frac{1}{2} \langle \pi^* H_1, \Theta \wedge \Theta \rangle + \frac{1}{6} \langle \pi^* H_0, \Theta \wedge \Theta \wedge \Theta \rangle$$

(but $B_\alpha = B_{2\alpha} + \langle B_{1\alpha}, \Theta \rangle + \frac{1}{2} \langle B_{0\alpha}, \Theta \wedge \Theta \rangle$ (No π^*!))

$\longrightarrow$ $H_j \in \Omega^j(M; \Lambda^{3-j}\mathfrak{t})$ for $j = 0, 1, 2, 3$

$\longrightarrow$ $\langle \cdot, \cdot \rangle$: the natural pairing $\mathfrak{t}^* \otimes \mathfrak{t} \longrightarrow \mathbb{R}$

$\longrightarrow$ $dH = 0 \quad \Rightarrow \quad dH_j + \langle H_{j-1}, F \rangle = 0$

$\mathbb{T}^n$ **action** $\&$ **Courant**

- Courant bracket:
  $[(v,\rho),(w,\lambda)]_H = [v,w] + \Big\{\mathcal{L}_v\lambda - \mathcal{L}_w\rho - \frac{1}{2}\,\mathrm{d}(\imath_v\lambda - \imath_w\rho) + \imath_v\imath_w H\Big\}$

- $v = v_M + \langle K, f\rangle$ and $\rho = \rho_M + \langle \phi, \Theta\rangle$
  $\mathcal{L}_K v = 0$ and $\mathcal{L}_K \rho = 0$, $\Rightarrow f \in \Omega^0(M,\mathfrak{t})$ and $\phi \in \Omega^0(M,\mathfrak{t}^*)$. ($\mathbb{T}^n$-invariant section of TX can be written as an element $(v_M, f) \in TM \oplus \mathfrak{t}$, while a $\mathbb{T}^n$-invariant section of TX^* can be written as $(\rho_M, \phi) \in T^*M \oplus \mathfrak{t}^*$.)

- Courant with $\mathbb{T}^n$ action:

$$[(v_M, f;\rho_M,\phi),\ (w_M, g;\lambda_M,\omega)]_H = [(v_M;\rho_M),\ (w_M;\lambda_M)]_{H_3} +$$

$$\Big(\underbrace{0}_{\text{vector}}, \underbrace{\mathcal{L}_{v_M} g - \mathcal{L}_{w_M} f}_{\text{function}}; \underbrace{\langle\omega, df\rangle - \langle\phi, dg\rangle - d(\langle\omega, f\rangle - \langle\phi, g\rangle)/2}_{1-\text{form}}, \underbrace{\mathcal{L}_{v_M}\omega - \mathcal{L}_{w_M}\phi}_{\text{function}}\Big) +$$

$$\Big(0, \imath_{v_M}\imath_{w_M}F; \langle\omega, \imath_{v_M}F\rangle + \langle\imath_{v_M}F_\#, g\rangle - \langle\imath_{w_M}F_\#, f\rangle - \langle\phi, \imath_{w_M}F\rangle, \imath_{v_M}\imath_{w_M}F_\#\Big) -$$

$$\Big(0, 0; \underline{\langle H_1, [f,g]\rangle}, \langle H_0, [f,g]\rangle\Big)$$

(don't like H_1 and H_0!)

where $F^I_\# := \imath(\frac{\partial}{\partial\theta_I})H = H^I_2 + (-H^{IJ}_1 \wedge \Theta_J + \frac{1}{2}\,H^{IJK}_0\Theta_J \wedge \Theta_K)$ and $dF^I_\# = 0$

Automorphisms of the bracket:

◇ Constant $O(n,n)$ transformations on $\mathfrak{t} \oplus \mathfrak{t}^*$:

$$S_{\mathfrak{t}}(X) = \left(\begin{array}{cc|cc} \mathbb{I} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & \mathbb{I} & 0 \\ 0 & C & 0 & D \end{array}\right) \left(\begin{array}{c} v \\ f \\ \hline \rho \\ \phi \end{array}\right)$$

◇ Generalized B-transforms

$$X \mapsto e^{\hat{B}} X = \Big(v, f + \imath_v U; \rho + \imath_v b^{\mathcal{B}} + \langle b, f \rangle + \langle \phi, U \rangle, \phi + \imath_v b\Big)^T$$

with closed two-form $b^{\mathcal{B}}$ and one-forms b and U.

- The automorphism (unobstructed T-duality):

$$[S_{\mathfrak{t}}(e^{\hat{B}} X), S_{\mathfrak{t}}(e^{\hat{B}} Y)] = S_{\mathfrak{t}}(e^{\hat{B}} [X, Y])$$

(one-forms b and U are coordinate dependent now)

Automorphisms with coordinate-dependent $O(n,n)$ $\longrightarrow$ "TWIST"

“TWIST” and PURE SPINORS

◇ $\Phi_\pm \quad \longmapsto \quad \Phi'_\pm = O^\pm \cdot \Phi_\pm$

◇ $O = e^{i\theta_c^\pm} \frac{1}{\sqrt{\det A}} e^{-y_{mn}\mathrm{d}x^m \wedge \mathrm{d}x^n} \, e^{a^m{}_n \mathrm{d}x^n \wedge \iota_{\partial_m}} \, e^{x_{mn}\mathrm{d}x^m \wedge \mathrm{d}x^n} = e^{i\theta_c^\pm} O_f$

O is a combination of

- B-transform
- a scaling transformation of (parts of) the metric $\quad \Rightarrow \quad$ a shift in dilaton
- a (pair of) $U(1)$ rotation(s)
- a change in the connection - twist of $\mathbb{T}^n$

For $\Phi'_\pm \quad \Rightarrow$
$$\begin{cases} \mathrm{d}\,(e^{3A}\Phi'_1) = 0 & \Leftrightarrow \quad \text{Gen. CY structure} \\ \mathrm{d}\,(e^{2A}\mathrm{Re}\Phi'_2) = 0 & \\ \mathrm{d}\,(e^{4A}\mathrm{Im}\Phi'_2) = R' & \end{cases}$$

◇ $\mathrm{d}(O_f)\,\Phi_1 = 0$

◇ New integrability defect from RR part $R = e^{4A} e^{-B} * \lambda(F)$:

$R' = \cos(\theta_c^+) O_f \, R + \sin(\theta_c^+)\mathrm{d}(e^{2A}O_f)\, e^{2A}\mathrm{Re}\Phi_2 + \cos(\theta_c^+)\mathrm{d}(O_f)\, e^{4A}\mathrm{Im}\Phi_2$

TWISTED TORUS BACKGROUNDS FROM TWIST DUALITY

- $\mathbb{T}^4 \times \mathbb{T}^2 \quad \Rightarrow \quad \mathbb{T}^2 \hookrightarrow M \xrightarrow{\pi} \mathbb{T}^4$
- $SU(3)$ structure after the twist:

$$J' = J_M + \frac{i}{2} g'_{z\bar{z}}\, \Theta \wedge \overline{\Theta}$$

$$\Omega' = \sqrt{g'}\, \omega_M \wedge \Theta$$

- PS equations $\Rightarrow$ conditions on curvature $\pi^* F = \mathrm{d}\Theta$:

$$\left.\begin{array}{l} F \wedge J_M = \overline{F} \wedge J_M = 0 \\ F \wedge \omega_M = 0 \end{array}\right\} \quad \Rightarrow \quad F = F^{2,0} + F^{1,1}_{-}$$

- Five solutions:

$(0,0,0,0,12,34)$	$M = N_3 \times N_3$	$b_2(M) = 8$
$(0,0,0,0,13,14)$	$M = S^1 \times M_5$	$b_2(M) = 9$
$(0,0,0,0,2\times 13, 14+23)$	$M = N_6^{(1)}$	$b_2(M) = 8$
$(0,0,0,0,13+42,14+23)$	$M = N_6^{(2)}$	$b_2(M) = 8$
$(0,0,0,0,0,14-23)$	$M = S^1 \times N_5$	$b_1(M) = 5$

ITERATING THE TWIST

◇ $S^1 \times M_5 \quad \Rightarrow \quad S^1 \hookrightarrow M \xrightarrow{\pi'} M_5$

◇ Conditions on curvature :

$$
\begin{aligned}
F \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) &= 0 \\
F \wedge (e^1 \wedge e^3 \wedge e^4 + e^1 \wedge e^5 \wedge e^6) &= 0
\end{aligned}
$$

◇ $M : (0, F, 0, 0, 13, 14) \cong (0, 0, 0, 12, 23, 14 - 35), \qquad b_1(M) = 3$

◇ Bianchi Identity

$$g_s \mathrm{d}F_3 = 2i\partial\bar{\partial}(e^{-2A}J) = \delta(D5) - \delta(O5)$$

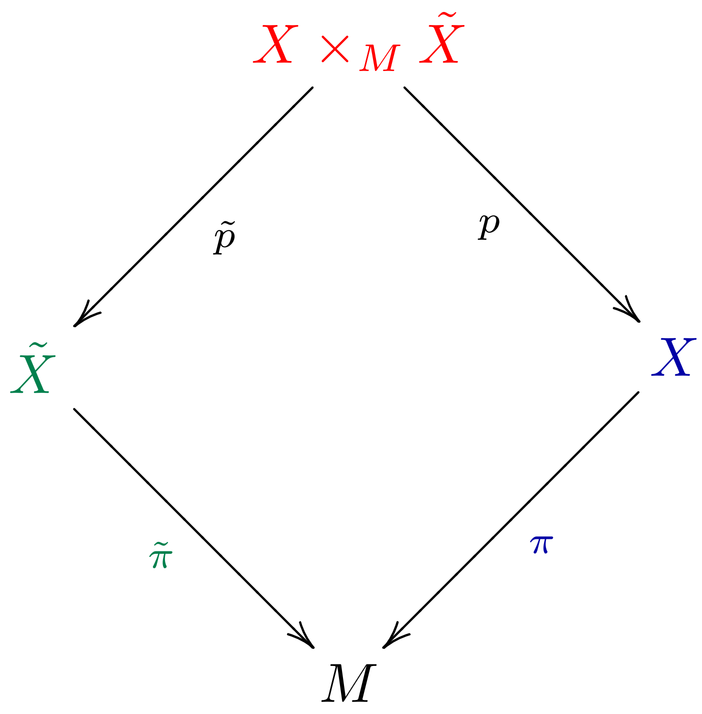
with intersecting sources.

- Can recover ALL solutions on iterations of torus bundles over tori (nilamnifolds)

T-duality - coord-independent $O(n,n)$ transformation in a background with n isometries v^i. Acts separately on NS and RR sectors - perturbative symmetry.

Preserves susy if $\mathcal{L}_{v^i}\Phi_\pm = 0$.

Topology change:



Correspondence space $Y = X \times_M \tilde{X}$:

- a circle bundle over X with first Chern class $\pi^*(c_1(\tilde{X}))$
- a circle bundle over $\tilde{X}$ with first Chern class $\tilde{\pi}^*(c_1(\tilde{X}))$

T-duality:

$$\pi_* H = c_1(\tilde{X}) \quad \tilde{\pi}_* \tilde{H} = c_1(X) \qquad \in H^2(M)$$

EXAMPLE: twisted torus (nilmanifold)

T duality: $\int H \leftrightarrow c_1(\tilde{X})$

$$\left.\begin{array}{l} T^3:\ (x^1, x^2, x^3) \\ H_3:\ N dx^1 \wedge dx^2 \wedge dx^3 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \tilde{X}_3:\ (de^1 = 0; de^2 = 0; de^3 = N e^1 \wedge e^2) \Leftrightarrow (0, 0, N \times 12) \\ \tilde{H}_3:\ 0 \end{array}\right.$$

$\tilde{X}_3 = G/\Gamma:\quad (x^1, x^2, x^3) \sim (x^1, x^2 + a, x^3) \sim (x^1, x^2, x^3 + b) \sim (x^1 + c, x^2, x^3 - Ncx^1))$

$\rightarrow$ nilpotent Twist

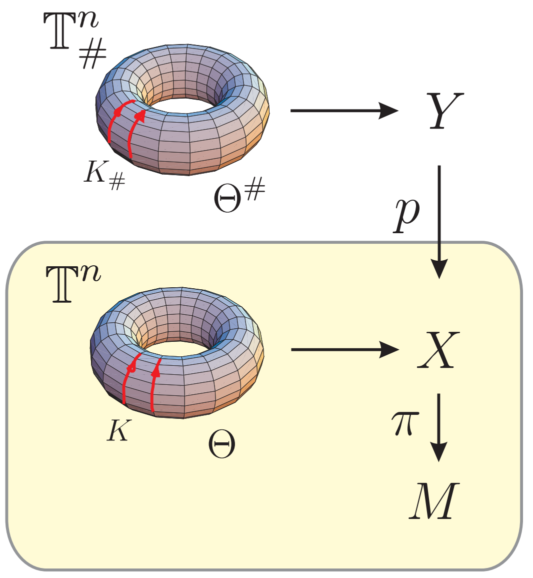
$\tilde{X}_3$ is an S^1 fibration over T^2 with $c_1 = N$

!! Choosing $B_2 = N x^1 dx^2 \wedge dx^3$ have two isometries. BUT... problems with the second T-duality: After the first T-duality, $ds^2_{\tilde{X}} = dx_1^2 + dx_2^2 + (dx_3 + N x_1 dx_2)^2$, and ∂_2 is no longer well-defined (due to x^1 being periodic - this not a local problem) !!

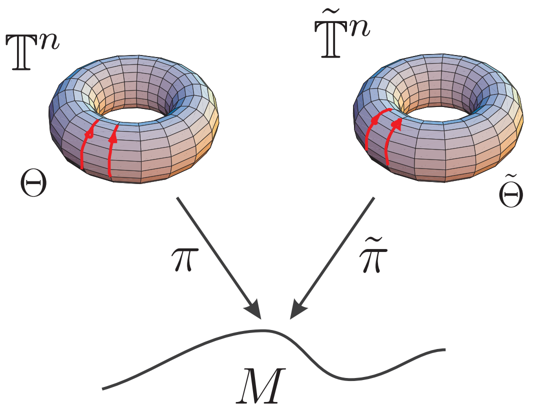
$\Rightarrow$ generalization of the correspondence space?

Action of O(n,n) is obstructed for $H_0 \neq 0$ & $H^1(M) \neq 0$!!!

Geometrization of H ($d\imath(\frac{\partial}{\partial\theta_I})H = 0 \Rightarrow p^*(F^I_\#) = d\Theta^I_\#$) gives



instead of: Y - correspondence space



with $\tilde{\Theta}^I = \Theta^I_\# + B_0^{IJ}\Theta_J$

Generalized correspondence space

- The gluing conditions for B_α compatible with the $\mathbb{T}^n$-invariance $\mathcal{L}(\frac{\partial}{\partial\theta_I})B_\alpha = 0$:

$$B_{0\alpha}^{IJ} - B_{0\beta}^{IJ} = m_{\alpha\beta}^{IJ}$$

  $\longrightarrow$ $\{m_{\alpha\beta}^{IJ}\}$ - skewsymmetric integral valued matrices defined on $M_{\alpha\beta}$ satisfying cocycle condition $m_{\alpha\beta} + m_{\beta\gamma} + m_{\gamma\alpha} = 0$

  $\longrightarrow$ $m_{\alpha\beta}^{IJ} \to 0 \qquad \Rightarrow \qquad B_0^{IJ}$ – globally defined smooth function; no obstructions for O(n,n) action

- Generalized correspondence space Y – affine $\mathbb{T}^n \times \mathbb{T}^n$-torus bundle over M:

  The gluing conditions for corresponding affine connection:

$$\begin{pmatrix} \tilde{\Theta}_\alpha \\ \Theta_\alpha \end{pmatrix} \equiv \begin{pmatrix} d\tilde{\theta}_\alpha + B_{1\alpha} \\ d\theta_\alpha + A_\alpha \end{pmatrix} = \begin{pmatrix} \mathbb{I} & m_{\alpha\beta} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \tilde{\Theta}_\beta \\ \Theta_\beta \end{pmatrix}$$

- $\Theta_{\#}^I|_{Y_\alpha} = \tilde{\Theta}_\alpha^I - B_{0\alpha}^{IJ}\,\Theta_J$ – a globally well defined 1-form on the total space of the affine torus bundle

Obstructed T-duality $\leftrightarrow$ **“non-geometric backgrounds”**

- Courant bracket as the algebra of sections of the generalised tangent bundle E:

$$0 \to T^*X \to E \to TX \to 0$$

$\rightarrow$ Choice of $B \Rightarrow$ identification with $T \oplus T^*$

$$(v + \xi)|_{X_\alpha} = v_\alpha + (\hat{\xi}_\alpha + i_v B_\alpha).$$

$\rightarrow$ On two-foold intersection $X_\alpha \cap X_\beta$:

$$\begin{cases} B_\alpha = B_\beta + dA_{\alpha\beta} \\ x_\alpha + \hat{\xi}_\alpha = x_\beta + \hat{\xi}_\beta - i_X dA_{\alpha\beta} \end{cases}$$

$\rightarrow$ Courant on E $\Rightarrow$ **twisted** Courant on $T \oplus T^*$

$\rightarrow$ Global spinor on E: $e^{B_\alpha}\Phi_\alpha = e^{B_\beta}\Phi_\beta$

Generalised tangent bundle E

- Spin connection $\omega_{ab} = \frac{1}{2}\left(f_{abc} + f_{acb} - f_{bca}\right) e^c$ is defined by:

$$[\tilde{v}_a, \tilde{v}_b] = f^c{}_{ab}\tilde{v}_c \ , \qquad \Leftrightarrow \qquad de^a = -\tfrac{1}{2} f^a{}_b \wedge e^b = -\tfrac{1}{2} f^a{}_{cb} e^c \wedge e^b \ .$$

- Passing to E (local basis): $E_a = \tilde{v}_a - \imath_{\tilde{v}_a} B, \qquad E^a = e^a$

- Restriction of the Courant bracket to the basis:

$$[E_a, E_b] = f^c{}_{ab}\, E_c - H_{abc}\, E^c$$
$$[E_a, E^b] = -f^b{}_{ac}\, E^c$$
$$[E^a, E^b] = 0$$

$$H = \tfrac{1}{6} H_{abc} e^a \wedge e^b \wedge e^c = \tfrac{1}{2}\left(\tilde{v}_c(B_{ab}) - f^d{}_{ab} B_{cd}\right) e^a \wedge e^b \wedge e^c \qquad (\tilde{v}_b(e^a) = -\tfrac{1}{2} f^a{}_{bc} e^c).$$

- PS equations lift to E

$$\rightarrow \qquad (d - H)\Phi \qquad \Leftrightarrow \qquad d\Phi_\alpha = 0$$

- T-duality preserves the form of the PS equations:

$$e^{B^{(2)}}\, \Phi_0^{\pm} \mapsto e^{\beta}\, \tilde{\Phi}_0^{\pm}$$
$$e^B d(e^{-B}\Phi_0) = d\Phi_0 - H \wedge \Phi_0 \mapsto e^{\beta} d(e^{-\beta}\tilde{\Phi}_0) = d\tilde{\Phi}_0 - Q \llcorner \tilde{\Phi}_0$$

Example: T-duality (action on gen. tangent space)

- $ds^2 = ds^2_M + \Theta^I\Theta^I, \qquad B = \frac{1}{2}B_{\mu\nu}\,dx^\mu \wedge dx^\nu + B_{\mu I}dx^\mu \wedge \Theta^I + \frac{1}{2}B_{IJ}\Theta^I \wedge \Theta^J$
- Torus fibration: $\Theta^I = d\theta^I + A^I$
- Basis on E :

$$\begin{pmatrix} E^{a'} \\ E^I \\ E_{a'} \\ E_I \end{pmatrix} = \left(\begin{array}{cc|cc} e_\mu^{a'} & 0 & & \\ A_\mu^I & \mathbb{I} & & \\ \hline -(B_{a'\mu} + B_{a'I}A_\mu^I) & -B_{a'I} & E_{a'}^\mu & -E_{a'}^\mu A_\mu^I \\ -(B_{I\mu} + B_{IJ}A_\mu^J) & -B_{IJ} & 0 & \mathbb{I} \end{array}\right) \cdot \begin{pmatrix} dx^\mu \\ d\theta^I \\ \partial_\mu \\ \partial_I \end{pmatrix}$$

- Two T-dualities ($I, J = 1, 2$) give:

$$T^T_{\{I\}} \cdot \left(\begin{array}{cc|cc} [e_\mu^{a'}]_{4\times 4} & 0 & & \\ A_\mu^I & \mathbb{I}_{2\times 2} & & \\ \hline -\hat{B}_{a'\mu} & -B_{a'I} & E_{a'}^\mu & -A_{a'}^I \\ -\tilde{B}_{I\mu} & -B_{IJ} & 0 & \mathbb{I} \end{array}\right) \cdot T_{\{I\}} \Leftrightarrow \left(\begin{array}{cc|cc} e_\mu^{a'} & 0 & 0 & 0 \\ (\epsilon\tilde{B})_\mu^I & -(\epsilon\cdot\epsilon)_I^J & 0 & -(\epsilon B\epsilon)^{IJ} \\ \hline -\hat{B}_{a'\mu} & -(\epsilon A)_{a'I} & E_a^\mu & -(\epsilon B)_{a'}^I \\ -(\epsilon A)_{I\mu} & 0 & 0 & -(\epsilon\cdot\epsilon)_J^I \end{array}\right)$$

- $\hat{B}_{\mu\nu}$ - invariant; $\qquad A_\mu^I \leftrightarrow \tilde{B}_{I\mu} = B_{I\mu} + B_{IJ}A_\mu^J; \qquad B_{IJ} \leftrightarrow (\epsilon B\epsilon)^{IJ} = \beta^{IJ}$

β - transform and Courant bracket

- Gluing with β transform:

$$(X+\xi)|_{X_\alpha} = (X_\alpha + \xi \lrcorner \beta_\alpha) + \hat{\xi}$$

Cannot identify with $T \oplus T^*$!!! (unless ...)

- New basis:

$$E_a = \tilde{v}_a\,, \qquad E^a = e^a + (e^a \llcorner \beta) = e^a + \beta^{ab}\tilde{v}_b$$

- New algebra:

$$[E_a, E_b] = f^c{}_{ab}\, E_c$$

$$[E_a, E^b] = -f^b{}_{ac}\, E^l + Q^{bc}{}_a\, E_c$$

$$[E^a, E^b] = Q^{ab}{}_c\, E^c + R^{abc}\, E_c$$

- Generalized spin connection:

$\longrightarrow$ $$\omega_{ab} = \tfrac{1}{2}\,(f_{abc} + f_{acb} - f_{bca})\, e^c$$

$\longrightarrow$ $$Q^{ab}{}_c = \tilde{v}_c(\beta^{ab}) + \beta^{ad} f^b{}_{cd} - \beta^{bd} f^a{}_{cd}$$

$\longrightarrow$ $$R^{abc} = \beta^{ad}\,\tilde{v}_d(\beta^{bc}) - \beta^{bd}\,\tilde{v}_d(\beta^{ac}) + \beta^{ad}\beta^{be} f^c{}_{de}$$

- Background with isometries

$\rightarrow \quad \mathbb{T}^n \hookrightarrow X \xrightarrow{\pi} M$:

$$f^c{}_{ab} = \begin{cases} f^{c'}{}_{a'b'} & \text{for } a', b', c' \text{ - along the base} \\ f^I{}_{a'b'} & \text{for } I, J \text{ - along the fibre } \mathbb{T}^d, \quad ([\tilde{v}_I, \tilde{v}_J] = 0) \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow \quad \beta = \frac{1}{2}\, \beta^{IJ} \imath_I \wedge \imath_J$

- $R = 0$ and $Q = d\,\beta$
- For constant β along $\mathbb{T}^d$ – β–transform is an auto-morphism of the Courant bracket
- Local $GL(n)$ transformation can take $\beta \rightarrow B$ ($Q \rightarrow 0$) but ...

$\rightarrow$ If $H^1(M, \mathbb{Z}) \neq 0$ the transformation is not single valued!

$\rightarrow$ If $H^1(M, \mathbb{Z}) = 0$, Q can be shifted away

General O(n,n)

- The spinor:

$$\Phi^{\pm} = e^{A+B+\beta}\Phi^{\pm}_{(0)}$$

- The algebra:

$$[E_m, E_n] = f^l{}_{mn}\, E_l + H_{mnl}\, E^l$$
$$[E_m, E^n] = -\tilde{f}^n{}_{ml}\, E^l - \tilde{Q}^{nl}{}_m\, E_l$$
$$[E^m, E^n] = Q^{mn}_l\, E^l + R^{mnl}\, E_l$$

The conclusion

- For $H^1(M,\mathbb{Z}) \neq 0$, O(n,n) action is obstructed, and no restriction to $T \oplus T^*$ (of anything!) is possible. For $H^1(M,\mathbb{Z}) = 0$, can think of the O(n,n)-inv. algebra as a (twisted) Courant on E (i.e. choose $Q \to 0$).
- SUSY equations continue to hold!

NON-supersymmetric (incl. DE SITTER) Solutions

- The theory has source terms:

$$S_{bulk} - T_p \int \mathrm{d}^{p+1}x \; e^{-\phi} \sqrt{|i^*[g_{10}] + \mathcal{F}|} \,, \qquad T_p^2 = \frac{\pi}{\kappa^2}(4\pi^2\alpha')^{3-p}$$

- for SUSY branes
  - use calibrations to replace the volume form on the brane worldvolume by the pullback of the non-integrable pure spinor

$$\left(i^*[\mathrm{Im}\Phi_-] \wedge e^{\mathcal{F}}\right) = \frac{|a|^2}{8} \sqrt{|i^*[g_{10}] + \mathcal{F}} \mathrm{d}^{\Sigma}x$$

  - keeping this condition does not allow for SUGRA de Sitter vacua

- An essential property of SUSY branes: $*\lambda(F)$ is trivial in $(\mathrm{d} - H)$-cohomology:

$$\mathrm{d}_H(e^{3A-\phi}\mathrm{Im}\Phi_-) = \frac{|a|^2}{8} e^{3A} * \lambda(F)$$

- Keep this property for SUSY-breaking sources - define a odd polyform (in IIA):

$$X = \sqrt{|g_4|}\, \mathrm{d}^4x \wedge X_- = \sqrt{|g_4|}\, \mathrm{d}^4x \wedge X_-$$

$$X_- = \frac{8}{||\Phi_-||}\Big(\alpha_0 \Phi_- + \overline{\alpha}_0 \overline{\Phi}_- + \alpha_{mn}\gamma^m \Phi_- \gamma^n + \overline{\alpha}_{mn}\gamma^m \overline{\Phi}_- \gamma^n$$

$$+\alpha^L_m \gamma^m \Phi_+ + \overline{\alpha}^L_m \gamma^m \overline{\Phi}_+ + \alpha^R_n \Phi_+ \gamma^n + \overline{\alpha}^R_n \overline{\Phi}_+ \gamma^n\Big)$$

such that

$$(\mathrm{d} - H)\mathrm{Im}X_- = c_0\ g_s * \lambda(F)$$

the RR-flux EOMs are automatic.

- use X_- to ”calibrate” the non-susy source

$$\left(i^*[\mathrm{Im}X_-] \wedge e^{\mathcal{F}}\right) = \frac{|a|^2}{8}\sqrt{|i^*[g_{10}] + \mathcal{F}}\mathrm{d}^\Sigma x$$

- Can satisfy all EOM and BI!

CONCLUSIONS

- Generalised Complex Geometry appears as the unifying framework for describing flux backgrounds

- – $\mathcal{N}=1$ vacua are Generalised Calabi Yau manifolds
  – RR $*\lambda(F)$ trivial in $(\mathrm{d}-H)$ - cohomology $\leftrightarrow$ second pure spinor
  – compatibility of two pure spinors

- Geometry without supersymmetry (... with "integrability")

- First order equations for non-susy vacua from pure spinors

$$\begin{cases} (\mathrm{d}-H)\,\mathrm{Re}X_- = 0\,, \\ (\mathrm{d}-H)\,\mathrm{Im}X_- = c_0\; g_s\, e^{-B} * \lambda(F) \end{cases}$$

- Integrability of GCS as a stability condition $(\mathrm{d}-H)X_+ = \imath_v X_+ + \rho\wedge X_+$

- Lack of compatibility $\langle X_+, \gamma X_-\rangle = m * \mathrm{Vol}_6$ ($m \neq 0!$) as susy breaking