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Interaction for higher spin gauge fields and Noether's procedure

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Plan:

1. *Introduction and Motivation*
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3. S_1 - S_2 -S_3 Interaction
4. Generating function for HS cubic interactions
5. Conclusion

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1. Introduction and Motivation

- The construction of interacting ***higher spin gauge field*** theories (HSF) has always been considered an important task. The construction of (self)interacting HSF theories was always in the center of attention during the last thirty years

- Particular attention caused the holographic duality between the *O(N)* sigma model in d=3 space and HSF gauge theory living in the AdS_4. This case of holography is singled out by the existence of two conformal points of the boundary theory and the possibility to describe them by the same HSF gauge theory with the help of *spontaneously breaking of higher spin gauge symmetry and mass generation by a corresponding Higgs mechanism*.

- *Does AdS/CFT works correctly on the level of loop diagrams in the general case and is it possible to use this correspondence for real reconstruction of unknown local interacting theories on the bulk from more or less well known conformal field theories on the boundary side?*

All these complicated physical tasks necessitate *quantum loop* calculations for HSF field theory and therefore information about manifest, off-shell and Lagrangian formulation of possible interactions for HSF.

1. Introduction and Motivation

- **Important physical tasks (AdS/CFT,…)**

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  Quantum loop calculations for HS

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  Lagrangian formulation of possible interactions for HSF.

- **String Theory includes a plethora of higher-spin excitations, whose detailed behavior is largely unknown.**

- **There are consistent equations for HSF in AdS (Vasiliev)**

- **Interacting Lagrangian for HS** ⟹ **???**

Gauge invariance

Unique Cubic Interaction
for arbitrary HS fields!!!

Power Expansion of Lagrangian and Gauge transformation

$$\delta = \delta^{(0)} + \delta^{(1)} + \ldots$$

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \ldots$$

Gauge Symmetry

$$\delta \mathcal{L} = 0,$$

Noether Equation

$$\sum (\delta_i^{(0)} + \delta_i^{(1)} + \ldots)(\mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \ldots) = 0,$$

The beauty of Noether's procedure :

2. Exercises on spin 1- field coupling with the HS gauge fields

- We start this section constructing the well known interaction of the electromagnetic field in flat **D** -dimensional space-time with the linearized spin two field.
- Noether’s procedure

$$L_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} (\partial A)^2,$$

$$\delta^0_\varepsilon h^{(2)\mu\nu}(x) = 2\partial^{(\mu} \varepsilon^{\nu)}(x) = \partial^\mu \varepsilon^\nu(x) + \partial^\nu \varepsilon^\mu(x), \qquad \delta^1_\varepsilon A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + C \varepsilon^\rho \partial_\mu A_\rho$$

$$\delta^1_\varepsilon L_0(A_\mu) + \delta^0_\varepsilon L_1(A_\mu, h^{(2)}_{\mu\nu}) = 0$$

Noether’s procedure regulates the relation between gauge symmetries of different spin fields → $C = 1$

- **Solution of Noether's Equation**

Reparametrization

$-\mathcal{L}_\varepsilon A_\mu$

Field dependent Gauge Variation

$$\delta^1_\varepsilon A_\mu = \varepsilon^\rho F_{\mu\rho}$$

$$= -\varepsilon^\rho \partial_\rho A_\mu - \partial_\mu \varepsilon^\rho A_\rho + \partial_\mu \left(\varepsilon^\rho (x) A_\rho (x) \right),$$

$$[\delta^1_\eta, \delta^1_\varepsilon] A_\mu = \delta^1_{[\eta,\varepsilon]} A_\rho + \partial_\mu \left(\varepsilon^\rho \eta^\sigma F_{\rho\sigma}(x) \right)$$

$$\delta^1_\varepsilon L_0(A_\mu) = \partial^{(\mu} \varepsilon^{\nu)} F_{\mu\rho} F_\nu^{\ \rho} - \frac{1}{4} \partial_\alpha \varepsilon^\alpha F_{\mu\nu} F^{\mu\nu},$$

$$L_1(A_\mu, h^{(2)}_{\mu\nu}) = \frac{1}{2} h^{(2)\mu\nu} \Psi^{(2)}_{\mu\nu},$$

Energy-Momentum Tensor

$$\Psi^{(2)}_{\mu\nu} = -F_{\mu\rho} F_\nu^{\ \rho} + \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma},$$

- Now we turn to the first nontrivial case of the vector field interaction with a spin four gauge field
- Starting variation

$$\delta^1_\varepsilon A_\mu = \varepsilon^{\rho\lambda\sigma}(x)\partial_\rho\partial_\lambda F_{\mu\sigma}. \qquad \delta^0_\varepsilon h^{\mu\rho\lambda\sigma} = 4\partial^{(\mu}\varepsilon^{\rho\lambda\sigma)}, \quad \delta^0_\varepsilon h^{\rho\lambda\sigma}_\rho = 2\varepsilon^{\lambda\sigma}_{(1)}.$$

Notations

$$\varepsilon^{\mu\nu\dots}_{(1)} = \partial_\lambda\varepsilon^{\lambda\mu\nu\dots}, \quad \varepsilon^{\mu\dots}_{(2)} = \partial_\nu\partial_\lambda\varepsilon^{\nu\lambda\mu\dots}, \quad \dots$$

$$\delta^1_\varepsilon L_0 = -\partial^{(\mu}\varepsilon^{\rho\lambda\sigma)}\partial_{(\rho}F^\nu_\mu\partial_\lambda F_{\sigma)\nu} + \frac{1}{4}\varepsilon^{\lambda\sigma}_{(1)}\partial^\nu F_{\mu\lambda}\partial^\mu F_{\nu\sigma} + \frac{1}{4}\varepsilon^{\lambda\sigma}_{(1)}\partial_\lambda F_{\mu\nu}\partial_\sigma F^{\mu\nu}$$

$$+\partial^{(\mu}\varepsilon^{\nu)}_{(2)}F_{\mu\sigma}F^\sigma_\nu - \frac{1}{4}\varepsilon_{(3)}F_{\mu\nu}F^{\mu\nu} - \partial_\lambda(\varepsilon^{\lambda\sigma}_{(1)}F_{\mu\sigma})\partial_\nu F^{\nu\mu} - \frac{1}{4}\varepsilon^{\lambda\sigma}_{(1)}\partial^\mu F_{\mu\lambda}\partial^\nu F_{\nu\sigma} - \frac{1}{2}\partial^\rho\varepsilon^{\nu\lambda\sigma}\partial_\lambda F_{\sigma\rho}\partial^\mu F_{\mu\nu}$$

Field Redefinition

$$A_\mu \to A_\mu - \frac{1}{2}\partial_\lambda(h^{\alpha\lambda\sigma}_\alpha F_{\mu\sigma}) - \frac{1}{8}h^\alpha_{\alpha\mu\sigma}\partial_\beta F^{\beta\sigma}.$$

Variation Modification

$$\delta^1_\varepsilon A_\mu = \varepsilon^{\rho\lambda\sigma}\partial_\rho\partial_\lambda F_{\mu\sigma} + \frac{1}{2}\partial_\rho\varepsilon_{\mu\lambda\sigma}\partial^\lambda F^{\sigma\rho}.$$

Integrable Part

$$L_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\nu\alpha\beta}) = \frac{1}{4}h^{(4)\mu\nu\alpha\beta}\Psi^{(4)}_{\mu\nu\alpha\beta} + \frac{1}{2}h^{(2)\mu\nu}\Psi^{(2)}_{\mu\nu},$$

4-1-1 Interaction

$$\delta^1_\varepsilon L_0(A_\mu) + \delta^0_\varepsilon L_1(A_\mu, h^{(2)}_{\mu\nu}, h^{(4)}_{\mu\nu}) = 0$$

$$L_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\nu\alpha\beta}) = \frac{1}{4} h^{(4)\mu\nu\alpha\beta} \Psi^{(4)}_{\mu\nu\alpha\beta} + \frac{1}{2} h^{(2)\mu\nu} \Psi^{(2)}_{\mu\nu},$$

$$\Psi^{(4)}_{\mu\nu\alpha\beta} = \partial_{(\alpha} F^{\rho}_{\mu} \partial_{\beta} F_{\nu)\rho} - \frac{1}{2} g_{(\mu\nu} \partial^{\lambda} F_{\alpha\sigma} \partial^{\sigma} F_{\beta)\lambda} - \frac{1}{2} g_{(\mu\nu} \partial_{\alpha} F^{\sigma\rho} \partial_{\beta)} F_{\sigma\rho}.$$

$$\Psi^{(2)}_{\mu\nu} = -F_{\mu\rho} F^{\rho}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma},$$

$$\delta^1 A_\mu = \varepsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma} + \frac{1}{2} \partial_\rho \varepsilon_{\mu\lambda\sigma} \partial^\lambda F^{\sigma\rho},$$

$$\delta^0 h^{(4)\mu\nu\alpha\beta} = 4\partial^{(\mu} \varepsilon^{\nu\alpha\beta)}, \delta^0_\varepsilon h^{\mu\alpha\beta}_{\mu} = 2\varepsilon^{\alpha\beta}_{(1)},$$

$$\delta^0 h^{(2)\mu\nu} = 2\partial^{(\mu} \varepsilon^{\nu)}_{(2)}, \delta^0 h^{(2)\mu}_{\mu} = 2\varepsilon_{(3)}.$$

an additional spin two field coupling !!

Spin one gauge field couplings to the higher spin gauge fields

$$\delta^1_\varepsilon A_\mu = \varepsilon_\ell^{\mu_1 \cdots \mu_{l-1}} \nabla_{\mu_1} \ldots \nabla_{\mu_{l-2}} F_{\mu_{l-1}\mu},$$

Starting variation

$$\delta^0_\varepsilon h^{(\ell)\mu_1 \cdots \mu_l} = l \nabla^{(\mu_l} \varepsilon_\ell^{\mu_1 \mu_2 \cdots \mu_{l-1})}, \quad \delta^0_\varepsilon h_\alpha^{(\ell)\alpha\mu_1 \cdots \mu_{l-2}} = 2\varepsilon_{\ell(1)}^{\mu_1 \cdots \mu_{l-2}}$$

$$\delta^1_\varepsilon L_0(A_\mu) + \delta^0_\varepsilon L_1(A_\mu, h^{(2)\mu_1\mu_2}, \ldots., h^{(\ell)\mu_1 \cdots \mu_l}) = 0$$

$$\delta^1_\varepsilon L_0 = \sum_{m=1}^{\ell/2} \binom{\ell-m-1}{m-1} \left(-\nabla^{(\mu_{2m}} \varepsilon^{\mu_1 \ldots \mu_{2m-1})}_{\ell(l-2m)} \Psi_{\mu_1 \ldots \mu_{2m}}(A_\mu)\right)$$

$$+\sum_{m=1}^{\ell/2} \binom{\ell-m-1}{m-1} \frac{m-1}{m} \nabla_{\mu_{m+1}} \ldots \nabla_{\mu_{2m-2}} (\nabla_\nu \varepsilon^{\mu_1 \ldots \mu_{2m-2}}_{\ell(l-2m)\mu} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-1}} F^{\nu}_{\mu_m}) \nabla_\alpha F^{\alpha\mu}$$

$$+\sum_{m=1}^{\ell/2} \binom{\ell-m-1}{m-1} \frac{m-1}{2m} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-3}} (\varepsilon^{\mu_1 \ldots \mu_{2m-3}}_{\ell(l-2m+1)\mu} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-2}} \nabla^\nu F_{\nu\mu_{m-1}}) \nabla_\alpha F^{\alpha\mu}$$

$$-\sum_{m=1}^{\ell/2} \binom{\ell-m-1}{m-1} \frac{m-1}{l-2m+1} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} (\varepsilon^{\mu_1 \ldots \mu_{2m-2}}_{\ell(l-2m+1)} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-2}} F_{\mu_{m-1}\mu}) \nabla_\alpha F^{\alpha\mu}$$

Field redefinition terms **Variation Modification**

Integrable terms

Interaction

$$L_1(A_\mu, h^{(2)}, h^{(4)}, \ldots, h^{(\ell)}) = \sum_{m=1}^{\ell/2} \frac{1}{2m} h^{(2m)\mu_1\ldots\mu_{2m}} \Psi^{(2m)}_{\mu_1\ldots\mu_{2m}}(A_\mu)$$

$$\Psi_{\mu_1\ldots\mu_{2m}}(A_\mu) = (-1)^m \Big(-\nabla_{\mu_1}\ldots\nabla_{\mu_{m-1}} F^{\nu}_{\mu_m} \nabla_{\mu_{m+1}}\ldots\nabla_{\mu_{2m-1}} F_{\mu_{2m}\nu}$$

$$+\frac{m-1}{2} g_{\mu_1\mu_2} \nabla_{\mu_3}\ldots\nabla_{\mu_m} \nabla^{\alpha} F_{\mu_{m+1}\beta} \nabla_{\mu_{m+2}}\ldots\nabla_{\mu_{2m-1}} \nabla^{\beta} F_{\mu_{2m}\alpha}$$

$$+\frac{m}{4} g_{\mu_1\mu_2} \nabla_{\mu_3}\ldots\nabla_{\mu_{m+1}} F^{\rho\sigma} \nabla_{\mu_{m+2}}\ldots\nabla_{\mu_{2m}} F_{\rho\sigma} \Big)$$

$$\delta^1_\varepsilon A_\mu = \varepsilon_\ell^{\mu_1\ldots\mu_{l-1}} \nabla_{\mu_1}\ldots\nabla_{\mu_{l-2}} F_{\mu_{l-1}\mu}$$

$$+\sum_{m=1}^{\ell/2} \binom{\ell-m-1}{m-1} \frac{m-1}{m} \nabla_{\mu_{m+1}}\ldots\nabla_{\mu_{2m-2}} (\nabla_\nu \varepsilon^{\mu_1\ldots\mu_{2m-2}}_{\ell(l-2m)\mu} \nabla_{\mu_1}\ldots\nabla_{\mu_{m-1}} F^{\nu}_{\mu_m})$$

Final variation

Field Redefinition

$$A_\mu \to A_\mu - \sum_{m=1}^{\ell/2} \binom{\ell - m - 1}{m-1} \frac{m-1}{2(l-2m+1)} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-2}} (h_\alpha^{(2m)\alpha\mu_1\ldots\mu_{2m-2}} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-2}} F_{\mu_{m-1}\mu})$$

$$+\sum_{m=1}^{\ell/2} \binom{\ell - m - 1}{m-1} \frac{m-1}{4m} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m-3}} (h_{\alpha\mu}^{(2m)\alpha\mu_1\ldots\mu_{2m-3}} \nabla_{\mu_1} \ldots \nabla_{\mu_{m-2}} \nabla^\nu F_{\nu\mu_{m-1}})$$

The gauge invariant action for the spin ℓ gauge field coupled to the spin 1 gauge field includes gauge invariant actions of tower of all smaller even spin gauge fields coupled to the same vector field in the same way.

Formalism

The most elegant and convenient way of handling symmetric tensors is by contracting it with the s'th tensorial power of a vector a^{μ_i} and star contraction

$$h^{(s)}(z;a)=\sum_{\mu_i}\Big(\prod_{i=1}^{s}a^{\mu_i}\Big)h^{(s)}_{\mu_1\mu_2\ldots\mu_s}(z).$$

$$*_a=\frac{1}{(s!)^2}\prod_{i=1}^{s}\overleftarrow{\partial}_a^{\mu_i}\overrightarrow{\partial}_{\mu_i}^{a}.$$

$$Tr:h^{(s)}(z;a)\Rightarrow Trh^{(s-2)}(z;a)=\frac{1}{s(s-1)}\Box_a h^{(s)}(z;a),$$

$$Grad:h^{(s)}(z;a)\Rightarrow Gradh^{(s+1)}(z;a)=(a\nabla)h^{(s)}(z;a),$$

$$Div:h^{(s)}(z;a)\Rightarrow Divh^{(s-1)}(z;a)=\frac{1}{s}(\nabla\partial_a)h^{(s)}(z;a).$$

$$(a\nabla)f^{(m-1)}(z;a)*_a g^{(m)}(z;a)=-f^{(m-1)}(z;a)*_a\tfrac{1}{m}(\nabla\partial_a)g^{(m,n)}(z;a)$$

$$a^2f^{(m-2)}(a)*_a g^{(m)}(a^m)=f^{(m-2)}(a,b)*_a\tfrac{1}{m(m-1)}\Box_a g^{(m)}(a).$$

Duality relations

Higher spin gauge field theories: Free field Lagrangian and equation of motion

Free field Lagragian of higher spin *s* field

$$L_0(h^{(s)}(a)) = -\frac{1}{2} h^{(s)}(a) *_a \mathcal{F}^{(s)}(a) + \frac{1}{8s(s-1)} \Box_a h^{(s)}(a) *_a \Box_a \mathcal{F}^{(s)}(a),$$

Invariant in respect to gauge transformation

$$\delta_{(0)} h^{(s)}(z;a) = s(a\nabla)\varepsilon^{(s-1)}(z;a),$$

$$\Box_a \varepsilon^{(s-1)}(z;a) = 0, \qquad \Box_a^2 h^{(s)}(z;a) = 0.$$

Fronsdal's higher spin gauge potential has scaling dimension s-2 (zero for the s=2 graviton case).

Equation of motion

$$\delta L_0(h^{(s)}(a)) = -(\mathcal{F}^{(s)}(a) - \frac{a^2}{4}\Box_a \mathcal{F}^{(s)}(a)) *_a \delta h^{(s)}(a),$$

Free field equation of motion

$$\mathcal{F}^{(s)}(a) - \frac{a^2}{4}\Box_a \mathcal{F}^{(s)}(a) = 0$$

Or equivalent

$$\mathcal{F}^{(s)}(a) = 0,$$

where

$$\mathcal{F}^{(s)}(z;a) = \Box h^{(s)}(z;a) - s(a\nabla)D^{(s-1)}(z;a),$$

$$D^{(s-1)}(z;a) = Divh^{(s-1)}(z;a) - \frac{s-1}{2}(a\nabla)Trh^{(s-2)}(z;a),$$

$$\Box_a D^{(s-1)}(z;a) = 0.$$

Correct generalization of the Lorentz gauge condition in the case of s>2 could be only the so-called de Donder gauge condition

$$D^{(s-1)}(z;a) = 0.$$

In this gauge free field equation of motion is very simple

$$\Box h^{(s)}(z;a) = 0$$

Higher spin gauge field selfinteraction: General setting of the problem

Neother equation for higher spin symmetry

$$\delta L(h^{(s)}(a)) = \frac{\delta L(h^{(s)}(a)}{\delta h^{(s)}(a))} *_a \delta h^{(s)}(a) = 0,$$

For nonlinear Lagrangian

$$L(h^{(s)}(a)) = L_0(h^{(s)}(a)) + L_1(h^{(s)}(a)) + \ldots,$$

And nonlinear gauge transformation

$$\delta h^{(s)}(a) = \delta_{(0)} h^{(s)}(a) + \delta_{(1)} h^{(s)}(a) + \ldots.$$

First order equation

$$\delta_{(1)} L_0(h^{(s)}(a)) + \delta_{(0)} L_1(h^{(s)}(a)) = 0,$$

Noether equation for Higher spin selfinteraction

$$\delta_{(0)} L_1(h^{(s)}(a)) = (\mathcal{F}^{(s)}(a) - \frac{a^2}{4}\Box_a \mathcal{F}^{(s)}(a)) *_a \delta_{(1)} h^{(s)}(a),$$

$$\delta_{(1)} h^{(s)}(a) = \delta_{(1)} \tilde{h}^{(s)}(a) + a^2 \delta_{(1)} h^{(s-2)}(a) + (a^2)^2 \delta h^{(s-4)}(a) + \ldots,$$

After

$$\delta_{(1)} h^{(s)}(a) \Rightarrow \delta_{(1)} h^{(s)}(a) + \frac{a^2}{2(D+2s-2)} \Box_a \delta h^{(s)}_{(1)}(a),$$

Noether equation can be simplified to

$$\delta_{(0)} L_1(h^{(s)}(a)) = \mathcal{F}^{(s)}(a) *_a \delta h^{(s)}_{(1)}(a).$$

Selfinteraction for any spin

Ascribing the same dimensions to the free part of the Lagrangian that is quadratic in the fields and derivatives and to the interaction cubic in the fields, we arrive at the idea that:

*the number of derivatives in the interaction should be **S**.*

This type of interacting theories will behave in the same way as gravity.

*The number of derivatives in the first order variation should be **S**-1.*

For s=2 this consideration is of course in full agrement with the linearized expansion of the Einstein-Hilbert action.

We start from following initial Lagragian term

$$L_1^{initial}(h^{(s)}(a)) = \frac{1}{2s} h^{(s)}(b) *_b \Gamma^{(s)}(b,a) *_a h^{(s)}(a),$$

$$L_1^{initial}(h^{(s)}(a)) = \frac{1}{2s} h^{(s)}(b) *_b \Gamma^{(s)}(b,a) *_a h^{(s)}(a),$$

where

$$\Gamma^{(s)}(z;b,a) = \sum_{k=0}^{s} \frac{(-1)^k}{k!} (b\nabla)^{s-k} (a\nabla)^k (b\partial_a)^k h^{(s)}(z;a).$$

is a de Witt – Freedman curvature for spin *s* field.

Simple, but important formula

$$F(z)\nabla_\mu G(z)\nabla^\mu H(z) = \frac{1}{2}\left(\Box F(z)G(z)H(z) - F(z)\Box G(z)H(z) - F(z)G(z)\Box H(z)\right).$$

$$\delta_{(0)} D^{(s-1)}(z;a) = \Box \varepsilon^{(s-1)}(z;a).$$

Selfinteraction Lagrangian leading term

$$L_{(1)}^{leading}(h^{(s)}(z)) = \frac{1}{3s(s!)^3}\sum_{\alpha+\beta+\gamma=s}\binom{s}{\alpha,\beta,\gamma}\int_{z_1,z_2,z_3}\delta(z-z_1)\delta(z-z_2)\delta(z-z_3)$$

$$\left[(\nabla_1\partial_c)^\gamma(\nabla_2\partial_a)^\alpha(\nabla_3\partial_b)^\beta(\partial_a\partial_b)^\gamma(\partial_b\partial_c)^\alpha(\partial_c\partial_a)^\beta\right]h(a;z_1)h(b;z_2)h(c;z_3),$$

Corresponding first order on field gauge variation leading term

$$\delta_{(1)}^{leading}h^{(s)}(c;z) = \frac{1}{s!(s-1)!}\sum_{\alpha+\beta+\gamma=s}(-1)^\beta\binom{s-1}{\alpha-1,\beta,\gamma}\int_{z_1,z_2}\delta(z-z_1)\delta(z-z_2)$$

$$\left[(c\nabla_1)^\gamma(\nabla_2\partial_a)^{\alpha-1}(\nabla_1\partial_b)^\beta(\partial_a\partial_b)^\gamma(c\partial_b)^\alpha(c\partial_a)^\beta\right]\varepsilon(a;z_1)h(b;z_2).$$

where

$$\binom{s}{\alpha,\beta,\gamma} = \frac{s!}{\alpha!\beta!\gamma!}, \quad s = \alpha+\beta+\gamma.$$

Gauge transformation next to free part first order on gauge field

$$\delta_{(1)}^{leading} h^{(s)}(c;z) = \frac{1}{s!}\sum_{k=0}^{s-1} k! \binom{s-1}{k} \gamma^{(k)}_{(\varepsilon^{(s-1)})}(c,b;a) *_{a,b} (a\nabla)^{s-k-1}(c\partial_b)^{s-k} h^{(s)}(b),$$

where

$$\gamma^{(k)}_{(\varepsilon^{(s-1)})}(c,b;a) = \frac{k!}{(s-1)!}\sum_{i=0}^{k} \frac{(-1)^i}{i!}(c\nabla)^{k-i}(b\nabla)^i (c\partial_b)^i \left[(a\partial_b)^{s-1-k}\varepsilon^{(s-1)}(b)\right].$$

$$\gamma^{(k)}_{(\varepsilon^{(s-1)})}(c,b;a) = \Gamma^{(k)}(c,b;h_a^{(k)}(b)),$$

$$h_a^{(k)}(b) = \frac{k!}{(s-1)!}\left[(a\partial_b)^{s-1-k}\varepsilon^{(s-1)}(b)\right].$$

Spin 2 example – linearized gravity

Free field Lagrangian for spin two gauge field

$$L_0 = -\frac{1}{2}h^{\mu\nu}(\Box h_{\mu\nu} - 2\nabla_{(\mu}D_{\nu)}) + \frac{1}{4}h(\Box h - 2(\nabla D)),$$

where

$$D_\mu = (\nabla h)_\mu - \frac{1}{2}\nabla_\mu h, \quad h = h_\mu^{\ \mu}, \quad (\nabla D) = \nabla^\mu D_\mu.$$

Which is invariant under gauge transformation

$$\delta_{(0)} h_{\mu\nu} = 2\nabla_{(\mu}\varepsilon_{\nu)}.$$

Selfinteraction for spin two gauge field

Trilinear selfinteraction Lagrangian

$$L_1(h^{(2)}) = \frac{1}{2} h^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{\mu\nu}h^{\mu\nu} + h^{\alpha\mu}\nabla_\alpha h^{\beta\nu}\nabla_\beta h_{\mu\nu}$$

$$-\frac{1}{4}(\nabla D)h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^{\mu\nu}\nabla_\mu h D_\nu ,$$

Corresponding gauge transformation

$$\delta_{(1)}h_{\mu\nu} = \varepsilon^\rho\nabla_\rho h_{\mu\nu} + \gamma^{(1)\rho}_{(\mu}h_{\nu)\rho}, \qquad \gamma^{(1)}_{\mu\nu} = 2\nabla_{[\mu}\varepsilon_{\nu]}$$

And field redefinition

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{4}(hh_{\mu\nu} - \frac{1}{2(D-2)}h^2 g_{\mu\nu}).$$

Spin 4 case

Spin four free gauge field Lagrangian

$$L_0(h^{(4)}) = -\frac{1}{2} h^{\alpha\beta\gamma\delta} \mathcal{F}_{\alpha\beta\gamma\delta} + \frac{3}{2} \bar{h}^{\alpha\beta} \bar{\mathcal{F}}_{\alpha\beta},$$

Which is invariant in respect to gauge transformation

$$\delta_{(0)} h_{\alpha\beta\gamma\delta} = 4\nabla_{(\alpha} \varepsilon_{\beta\gamma\delta)}, \qquad \varepsilon_{\alpha\beta}{}^{\beta} = 0.$$

$$\mathcal{F}_{\alpha\beta\gamma\delta} = \Box h_{\alpha\beta\gamma\delta} - 4\nabla_{(\alpha} D_{\beta\gamma\delta)}, \quad \bar{\mathcal{F}}_{\alpha\beta} = \mathcal{F}_{\alpha\beta\gamma}{}^{\gamma} = \bar{h}_{\alpha\beta} - 2(\nabla D)_{\alpha\beta},$$

$$D_{\alpha\beta\gamma} = (\nabla h)_{\alpha\beta\gamma} - \frac{3}{2} \nabla_{(\alpha} \bar{h}_{\beta\gamma)}, \qquad \bar{h}_{\beta\gamma} = h_{\beta\gamma\alpha}{}^{\alpha},$$

$$D_{\alpha\beta}{}^{\beta} = 0, \quad \bar{h}_{\beta}{}^{\beta} = 0.$$

Trilinear selfinteraction Lagrangian for spin 4 gauge field

We introduce some "coordinate system" for classification of our interaction

$$L_1 = \sum_{\substack{i,j=0,1,2,3 \\ i+j\leq 3}} L_{ij}^{int}(h^{(4)}),$$

where

$$L_{ij}^{int}(h^{(4)}) \sim \nabla^{4-i}(D)^{i}(\overline{h}^{(4)})^{j}(h^{(4)})^{3-j-i}.$$

Interaction in de Donder gauge

$$L_{dD}^{int}(h^{(4)}) = \sum_{j=0}^{3} L_{0j}^{int}(h^{(4)}).$$

$\bar{h}$ \ D	0	1	2
0	hhh	Dhh	DDh
1	$\bar{h}hh$	$\bar{h}Dh$	$\bar{h}DD$
2	$\bar{h}\bar{h}h$	$\bar{h}\bar{h}D$	
3	$\bar{h}\bar{h}\bar{h}$		

$$L_{00}^{int}(h^{(4)}) = \frac{1}{8} h^{\alpha\beta\gamma\delta} h^{\mu\nu\lambda\rho} \Gamma^{(4)}_{\alpha\beta\gamma\delta,\mu\nu\lambda\rho} - \nabla^{\mu} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \nabla^{\beta} h^{\gamma\nu\lambda\rho} \nabla^{\delta} h_{\mu\nu\lambda\rho}$$

$$+ \frac{3}{4} \nabla^{\mu} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \nabla^{\nu} h^{\gamma\delta\lambda\rho} \nabla^{\beta} h_{\mu\nu\lambda\rho} \;\; + 3 \nabla^{\mu} \nabla_{\nu} h_{\alpha\beta\gamma\delta} h^{\alpha\nu\lambda\rho} \nabla^{\beta} \nabla_{\lambda} h^{\gamma\delta}_{\mu\rho},$$

$$L_{01}^{int}(h^{(4)}) = -\frac{3}{2} h_{\alpha\beta\gamma\delta} \nabla^{\alpha} \nabla^{\beta} h^{\gamma\nu\lambda\rho} \nabla^{\delta} \nabla_{\nu} \bar{h}_{\lambda\rho} - 3 h_{\alpha\beta\gamma\delta} h^{\delta}_{\nu\lambda\rho} \nabla^{\alpha} \nabla^{\beta} \nabla^{\nu} \nabla^{\lambda} \bar{h}^{\gamma\rho}$$

$$+ \frac{3}{2} \nabla_{\mu} h_{\alpha\beta\gamma\delta} \nabla^{\nu} \nabla^{\alpha} h^{\mu\beta\gamma\lambda} \nabla^{\delta} \bar{h}_{\nu\lambda} - \nabla^{\lambda} h^{\mu\alpha\beta\gamma} \nabla^{\rho} h^{\nu}_{\alpha\beta\gamma} \nabla_{\mu} \nabla_{\nu} \bar{h}_{\lambda\rho}$$

$$+ \frac{1}{4} h^{\mu\alpha\beta\gamma} h^{\nu}_{\alpha\beta\gamma} \nabla_{\mu} \nabla_{\nu} (\nabla\nabla\bar{h}),$$

$$L_{02}^{int}(h^{(4)}) = -\frac{3}{2} h_{\alpha\beta\gamma\delta} \nabla^{\alpha}\nabla^{\beta}\nabla^{\mu}\bar{h}^{\gamma\nu}\nabla^{\delta}\bar{h}_{\mu\nu} + \frac{3}{2} h_{\alpha\beta\gamma\delta}\nabla^{\alpha}\nabla^{\mu}\bar{h}^{\beta\nu}\nabla^{\gamma}\nabla_{\nu}\bar{h}^{\delta}_{\mu}$$

$$-\frac{3}{4}\nabla_{\mu}\nabla_{\nu} h_{\alpha\beta\gamma\delta}\nabla^{\alpha}\bar{h}^{\beta\nu}\nabla^{\gamma}\bar{h}^{\delta\mu} - \frac{3}{4} h_{\alpha\beta\gamma\delta}\nabla^{\alpha}\nabla^{\beta}\bar{h}^{\gamma\delta}(\nabla\nabla\bar{h})$$

$$-3\nabla_{\mu} h_{\alpha\beta\gamma\delta}\nabla_{\nu}\nabla^{\alpha}\bar{h}^{\beta\gamma}\nabla^{\delta}\bar{h}_{\mu\nu},$$

$$L_{03}^{int}(h^{(4)}) = \frac{3}{4}\nabla_{\mu}\nabla_{\nu}\bar{h}_{\alpha\beta}\nabla^{\alpha}\bar{h}^{\mu\lambda}\nabla^{\beta}\bar{h}^{\nu}_{\lambda} - \frac{3}{4}\nabla_{\mu}\bar{h}^{\nu\lambda}\nabla_{\nu}\bar{h}^{\mu}_{\lambda}(\nabla\nabla\bar{h}),$$

$$L_{10}^{int}(h^{(4)}) = 3\nabla_\alpha \nabla_\nu D_{\lambda\rho\beta} h^{\alpha\beta\gamma\delta} \nabla_\gamma h_\delta^{\nu\lambda\rho} + \frac{3}{2} \nabla^\rho D_{\alpha\beta\lambda} \nabla^\mu h^{\alpha\beta\gamma\delta} \nabla^\lambda h_{\rho\mu\gamma\delta}$$

$$-2\nabla^\delta D_{\nu\lambda\rho} \nabla^\nu h_{\alpha\beta\gamma\delta} \nabla^\lambda h^{\rho\alpha\beta\gamma} - \frac{3}{2} (\nabla D)^{\alpha\rho} \nabla^\mu h_{\alpha\beta\gamma\delta} \nabla^\beta h_{\rho\mu}^{\gamma\delta}$$

$$+\frac{1}{4} (\nabla D)^{\mu\nu} \nabla_\mu h_{\alpha\beta\gamma\delta} \nabla_\nu h^{\alpha\beta\gamma\delta} - \frac{1}{2} (\nabla D)^{\mu\nu} h_{\alpha\beta\gamma\delta} \nabla_\mu \nabla_\nu h^{\alpha\beta\gamma\delta}$$

$$-\nabla_\alpha (\nabla D)^{\mu\nu} h^{\alpha\beta\gamma\delta} \nabla_\mu h_{\nu\beta\gamma\delta} + \frac{3}{4} \nabla_\alpha (\nabla D)^{\mu\nu} h^{\alpha\beta\gamma\delta} \nabla_\beta h_{\mu\nu\gamma\delta},$$

$$
\begin{aligned}
L_{11}^{int}(h^{(4)}) = &-\frac{1}{2}\bar{h}^{\gamma\delta}\nabla_{\gamma}h_{\mu\nu\lambda\rho}\nabla_{\delta}\nabla^{\mu}D^{\nu\lambda\rho}+\frac{1}{2}\bar{h}^{\gamma\delta}\nabla_{\gamma}\nabla_{\delta}h_{\mu\nu\lambda\rho}\nabla^{\mu}D^{\nu\lambda\rho} \\
&+\frac{3}{4}\nabla^{\mu}\bar{h}^{\gamma\delta}h_{\mu\nu\lambda\rho}\nabla_{\gamma}\nabla^{\nu}D_{\delta}^{\lambda\rho}-\frac{3}{4}\nabla_{\mu}\bar{h}^{\gamma\delta}h^{\mu\nu\lambda\rho}\nabla_{\nu}\nabla_{\lambda}D_{\gamma\delta\rho} \\
&+\frac{9}{4}\nabla_{\mu}\bar{h}^{\gamma\delta}\nabla_{\rho}h_{\gamma\delta\nu\lambda}\nabla^{\lambda}D^{\mu\nu\rho}+3\bar{h}^{\gamma\delta}\nabla_{\rho}h_{\gamma\mu\nu\lambda}\nabla^{\mu}\nabla^{\nu}D_{\delta}^{\lambda\rho} \\
&+\frac{3}{2}\bar{h}^{\gamma\delta}\nabla_{\rho}h_{\gamma\mu\nu\lambda}\nabla^{\mu}\nabla_{\delta}D^{\nu\lambda\rho}-\frac{3}{2}\bar{h}^{\gamma\delta}\nabla_{\rho}\nabla_{\gamma}h_{\delta\mu\nu\lambda}\nabla^{\mu}D^{\nu\lambda\rho} \\
&-\frac{3}{4}(\nabla D)^{\gamma\delta}\nabla^{\mu}\bar{h}^{\nu\lambda}\nabla_{\gamma}h_{\delta\mu\nu\lambda}-\frac{3}{4}\nabla^{\mu}(\nabla D)^{\gamma\delta}\bar{h}^{\nu\lambda}\nabla_{\lambda}h_{\gamma\delta\mu\nu} \\
&+6\nabla^{\mu}\nabla_{\nu}(\nabla D)^{\gamma\delta}\bar{h}_{\gamma\lambda}h_{\delta\mu}^{\nu\lambda}+\frac{1}{4}\bar{h}^{\gamma\delta}D^{\mu\nu\rho}\nabla_{\gamma}h_{\delta\mu\nu\rho} \\
&-\frac{3}{8}\bar{h}^{\gamma\delta}D^{\mu\nu\rho}\nabla_{\mu}h_{\gamma\delta\nu\rho},
\end{aligned}
$$

$$L_{12}^{int}(h^{(4)}) = \frac{3}{4} D^{\mu\nu\rho} (\nabla \bar{h})^{\delta} \nabla_{\delta} \nabla_{\mu} \bar{h}_{\nu\rho} - \frac{9}{8} \nabla_{\gamma} \nabla_{\delta} D^{\mu\nu\rho} \bar{h}^{\gamma\delta} \nabla_{\mu} \bar{h}_{\nu\rho}$$

$$-3 D^{\mu\nu\rho} \bar{h}^{\gamma\delta} \nabla_{\gamma} \nabla_{\mu} \nabla_{\nu} \bar{h}_{\delta\rho} - 3 \nabla_{\mu} D_{\nu\rho\gamma} \nabla^{\nu} \bar{h}^{\gamma\delta} \nabla_{\delta} \bar{h}^{\mu\rho}$$

$$-\frac{3}{2} (\nabla D)^{\gamma\delta} \nabla_{\gamma} \bar{h}^{\mu\nu} \nabla_{\delta} \bar{h}_{\mu\nu} - \frac{3}{8} (\nabla D)^{\gamma\delta} \bar{h}^{\mu\nu} \nabla_{\gamma} \nabla_{\delta} \bar{h}_{\mu\nu}$$

$$+\frac{3}{2} \nabla^{\mu} (\nabla D)^{\gamma\delta} \nabla_{\gamma} \bar{h}_{\mu\nu} \bar{h}^{\nu}_{\delta} - 3 (\nabla D)^{\gamma\delta} \nabla_{\gamma} \nabla_{\mu} \bar{h}_{\delta\nu} \bar{h}^{\mu\nu}$$

$$-\frac{9}{4} (\nabla D)^{\gamma\delta} \nabla^{\mu} \bar{h}_{\gamma\nu} \nabla^{\nu} \bar{h}_{\delta\mu} + \frac{3}{2} \nabla^{\mu} (\nabla D)^{\gamma\delta} \nabla^{\nu} \bar{h}_{\gamma\delta} \bar{h}_{\mu\nu}$$

$$+\frac{3}{8} (\nabla D)^{\gamma\delta} \bar{h}_{\gamma\delta} (\nabla \nabla \bar{h}) + \frac{3}{8} \nabla^{\mu} \nabla^{\nu} (\nabla D)^{\gamma\delta} \bar{h}_{\gamma\mu} \bar{h}_{\delta\nu}$$

$$-\frac{3}{2} (\nabla D)^{\gamma\delta} \bar{h}_{\gamma\mu} \bar{h}^{\mu}_{\delta},$$

$$L_{20}^{int}(h^{(4)}) = 3D^{\alpha\beta\gamma}D^{\mu\nu\rho}\nabla_\mu\nabla_\nu h_{\rho\alpha\beta\gamma} - \frac{9}{4}D^{\alpha\beta\gamma}D^{\mu\nu\rho}\nabla_\alpha\nabla_\mu h_{\beta\gamma\nu\rho}$$

$$+3D^{\alpha\beta\gamma}\nabla^\rho D_\gamma^{\mu\nu}\nabla_\alpha h_{\beta\mu\nu\rho} + \frac{1}{2}(\nabla D)^{\gamma\delta}D^{\mu\nu\rho}\nabla_\gamma h_{\delta\mu\nu\rho},$$

$$L_{21}^{int}(h^{(4)}) = -\frac{3}{4}\bar{h}^{\gamma\delta}\nabla_\gamma D^{\mu\nu\rho}\nabla_\delta D_{\mu\nu\rho} + \frac{1}{8}(\nabla\nabla\bar{h})D^{\mu\nu\rho}D_{\mu\nu\rho}$$

$$+\frac{3}{4}(\nabla\bar{h})^\delta D^{\mu\nu\rho}\nabla_\mu D_{\delta\nu\rho} + \frac{9}{4}\bar{h}^{\gamma\delta}\nabla_\gamma D^{\mu\nu\rho}\nabla_\mu D_{\delta\nu\rho}$$

$$+3\bar{h}^{\gamma\delta}\nabla^\mu D_\gamma^{\nu\rho}\nabla_\nu D_{\delta\mu\rho} + 3\nabla^\mu\nabla_\nu\bar{h}^{\gamma\delta}D_\gamma^{\nu\rho}D_{\delta\mu\rho}$$

$$-\frac{3}{4}\bar{h}^{\gamma\delta}(\nabla D)^{\mu\nu}\nabla_\mu D_{\nu\gamma\delta} + 6\bar{h}^{\gamma\delta}\nabla^\mu(\nabla D)_\gamma^\nu D_{\delta\mu\nu}$$

$$+3\bar{h}^{\gamma\delta}(\nabla D)_\gamma^\nu(\nabla D)_{\delta\nu}.$$

$$\delta_{(1)} h_{\alpha\beta\gamma\delta} = \varepsilon^{\mu\nu\rho} \nabla_{\mu} \nabla_{\nu} \nabla_{\rho} h_{\alpha\beta\gamma\delta}$$

$$+3(\nabla_{\alpha} \varepsilon_{\rho}{}^{\mu\nu} - \nabla_{\rho} \varepsilon_{\alpha}{}^{\mu\nu}) \nabla_{\mu} \nabla_{\nu} h_{\beta\gamma\delta}{}^{\rho}$$

$$+3(\nabla_{\alpha} \nabla_{\beta} \varepsilon_{\nu\rho}{}^{\mu} - 2\nabla_{\alpha} \nabla_{\nu} \varepsilon_{\beta\rho}{}^{\mu} + \nabla_{\nu} \nabla_{\rho} \varepsilon_{\alpha\beta}{}^{\mu}) \nabla_{\mu} h_{\gamma\delta}{}^{\nu\rho}$$

$$+(\nabla_{\alpha} \nabla_{\beta} \nabla_{\gamma} \varepsilon_{\mu\nu\rho} - 3\nabla_{\alpha} \nabla_{\beta} \nabla_{\mu} \varepsilon_{\gamma\nu\rho} + 3\nabla_{\alpha} \nabla_{\mu} \nabla_{\nu} \varepsilon_{\beta\gamma\rho} - \nabla_{\mu} \nabla_{\nu} \nabla_{\rho} \varepsilon_{\alpha\beta\gamma}) h_{\delta}^{\mu\nu\rho}$$

$$+(trace\ terms\ O(g_{\alpha\beta}))$$

$$= \gamma^{(0)\mu\nu\rho}_{(\varepsilon^{(3)})} \nabla_{\mu} \nabla_{\nu} \nabla_{\rho} h_{\alpha\beta\gamma\delta} + 3\gamma^{(1)}_{(\varepsilon^{(3)})\alpha,\rho}{}^{\mu\nu} \nabla_{\mu} \nabla_{\nu} h_{\beta\gamma\delta}{}^{\rho}$$

$$+3\gamma^{(2)}_{(\varepsilon^{(3)})\alpha\beta,\nu\rho}{}^{\mu} \nabla_{\mu} h_{\gamma\delta}{}^{\nu\rho} + \gamma^{(3)}_{(\varepsilon^{(3)})\alpha\beta\gamma,\mu\nu\rho} h_{\delta}^{\mu\nu\rho}$$

$$+(trace\ terms\ O(g_{\alpha\beta})),$$

Cubic interactions for arbitrary spin: Leading terms

$$\mathcal{L}_I^{(0,0)}(h^{(s_1)}(a), h^{(s_2)}(b), h^{(s_3)}(c)) = \sum_{n_i} C^{s_1,s_2,s_3}_{n_1,n_2,n_3} \int dz_1 dz_2 dz_3 \delta(z_3 - z)\delta(z_2 - z)\delta(z_1 - z)$$

$$\hat{T}(Q_{12}, Q_{23}, Q_{31} \,|\, n_1, n_2, n_3) h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c)$$

where

$$\hat{T}(Q_{12}, Q_{23}, Q_{31} \,|\, n_1, n_2, n_3) = (\partial_a \partial_b)^{Q_{12}} (\partial_b \partial_c)^{Q_{23}} (\partial_c \partial_a)^{Q_{31}} (\partial_a \nabla_2)^{n_1} (\partial_b \nabla_3)^{n_2} (\partial_c \nabla_1)^{n_3}$$

Number of derivatives Δ → $n_1 + n_2 + n_3 = \Delta$

$$n_1 + Q_{12} + Q_{31} = s_1 \qquad Q_{12} = n_3 - \nu_3 \qquad n_i \geq \nu_i$$

$$n_2 + Q_{23} + Q_{12} = s_2 \qquad Q_{23} = n_1 - \nu_1 \qquad \nu_i = 1/2(\Delta + s_i - s_j - s_k)$$

$$n_3 + Q_{31} + Q_{23} = s_3 \qquad Q_{31} = n_2 - \nu_2$$

I,j,k are all different

$$\Delta_{min} = \max[s_i + s_j - s_k] = s_1 + s_2 - s_3$$

$$\nu_1 = s_1 - s_3 \qquad \nu_2 = s_2 - s_3 \qquad \nu_3 = 0$$

$$\sum_{ij} Q_{ij} = \Delta - \sum_i \nu_i$$

$$\sum_i \nu_i = 3/2\Delta - 1/2 \sum_i s_i$$

$$C^{s_1, s_2, s_3}_{n_1, n_2, n_3} = const \begin{pmatrix} \sum n_i - \sum \nu_i \\ n_1 - \nu_1, n_2 - \nu_2, n_3 - \nu_3 \end{pmatrix}$$

$$C^{s_1, s_2, s_3}_{n_1, n_2, n_3} = const \begin{pmatrix} s_3 \\ n_1 - s_1 + s_3, n_2 - s_2 + s_3, n_3 \end{pmatrix}$$

$$C^{s_1, s_2, s_3}_{Q_{12}, Q_{23}, Q_{31}} = const \begin{pmatrix} 1/2(\sum s_i - \Delta) \\ Q_{12}, Q_{23}, Q_{31} \end{pmatrix}$$

To derive the next term of interaction containing one deDonder expression we turn to Lagrangian formulation of the task and will solve Noether's equation

$$\sum_{i=1}^{3}\delta_i^{(1)}\mathcal{L}_i^0(h^{(s_i)}(a))+\sum_{i=1}^{3}\delta_i^{(0)}\mathcal{L}_I(h^{(s_1)}(a),h^{(s_2)}(b),h^{(s_3)}(c))=0,$$

where

$$\mathcal{L}_i^0(h^{(s_i)}(a))=-\frac{1}{2}h^{(s_i)}(a_i)*_{a_i}\mathcal{F}^{(s_i)}(a_i)+\frac{1}{8s_i(s_i-1)}\Box_{a_i}h^{(s_i)}(a_i)*_{a_i}\Box_{a_i}\mathcal{F}^{(s)}(a_i)$$

$$\delta_i^{(0)}h^{(s_i)}(a_i)=s_i(a_i\nabla_i)\varepsilon^{s_i-1}(z_i;a_i)$$

Shifting $\delta_i^{(1)}$ by trace term in the same way as in our last article we obtain the following functional equation:

$$\sum_{i=1}^{3}\delta_i^{(0)}\mathcal{L}_I(h^{(s_1)}(a),h^{(s_2)}(b),h^{(s_3)}(c))=0+O(\mathcal{F}^{(s_i)}(a_i))$$

$$C^{\{s_i\}}_{\{n_i\}}\hat{T}(Q_{ij}\,|\,n_i)[(a\nabla_1)\varepsilon^{(s_1-1)}h^{(s_2)}h^{(s_3)}+h^{(s_1)}(b\nabla_2)\varepsilon^{(s_2-1)}h^{(s_3)}+h^{(s_1)}h^{(s_2)}(c\nabla_3)\varepsilon^{(s_3-1)}]$$

$$=0+O(\mathcal{F}^{(s_i)}(a_i))$$

we see that all necessary information
we can find calculating these commutators

$$[\hat{T}(Q_{ij}\,|\,n_i),(a\nabla_1)]=Q_{31}\hat{T}(Q_{12},Q_{23},Q_{31}-1\,|\,n_1,n_2,n_3+1)-Q_{12}\hat{T}(Q_{12}-1,Q_{23},Q_{31}\,|\,n_1,n_2+1,n_3)$$

$$+n_1\hat{T}(Q_{12},Q_{23},Q_{31}\,|\,n_1-1,n_2,n_3)(\nabla_1\nabla_2)-Q_{12}\hat{T}(Q_{12}-1,Q_{23},Q_{31}\,|\,n_1,n_2,n_3)(\partial_b\nabla_2),$$

$$\nabla_1\nabla_2=\frac{1}{2}(\Box_3-\Box_2-\Box_1),\qquad \Box_i h^{(s_i)}(z_i;a_i)=\mathcal{F}^{(s_i)}(z_i;a_i)+s_iD^{(s_i-1)}$$

$$\Box_i\varepsilon^{(s_i-1)}(z_i;a_i)=\delta_i^{(0)}D^{(s_i-1)}$$

$$(\partial_{a_i}\nabla_i)=s_iD^{(s_i-1)}+1/2(a_i\nabla_i)\Box_{a_i}h^{(s_i)}(z_i;a_i),\quad a_i=a,b,c$$

Noether's Equation leads to

$$(n_2+1-\nu_2)C^{s_1,s_2,s_3}_{n_1,n_2+1,n_3}-(n_3+1-\nu_3)C^{s_1,s_2,s_3}_{n_1,n_2,n_3+1}=0$$

$$(n_3+1-\nu_3)C^{s_1,s_2,s_3}_{n_1,n_2,n_3+1}-(n_1+1-\nu_1)C^{s_1,s_2,s_3}_{n_1+1,n_2,n_3}=0$$

$$(n_1+1-\nu_1)C^{s_1,s_2,s_3}_{n_1+1,n_2,n_3}-(n_2+1-\nu_2)C^{s_1,s_2,s_3}_{n_1,n_2+1,n_3}=0$$

With the following solution

$$C^{s_1,s_2,s_3}_{n_1,n_2,n_3}=const\begin{pmatrix}\sum n_i-\sum \nu_i\\ n_1-\nu_1,n_2-\nu_2,n_3-\nu_3\end{pmatrix}$$

$$\sum\int \equiv \sum_{n_i}\int dz dz_1 dz_2 dz_3 \delta(z_1 - z)\delta(z_3 - z)\delta(z_2 - z)$$

$$\mathcal{L}_I^{(1,0)} = \sum\int \Big[\frac{s_1 n_1}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1 - 1, n_2, n_3) D^{(s_1-1)} h^{(s_2)} h^{(s_3)}$$

$$+ \frac{s_2 n_2}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1, n_2 - 1, n_3) h^{(s_1)} D^{(s_2-1)} h^{(s_3)}$$

$$+ \frac{s_3 n_3}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1, n_2, n_3 - 1) h^{(s_1)} h^{(s_2)} D^{(s_3-1)}\Big]$$

$$\mathcal{L}_I^{(2,0)} = \sum\int \Big[+ \frac{s_3 n_3 s_1 n_1}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1 - 1, n_2, n_3 - 1) D^{(s_1-1)} h^{(s_2)} D^{(s_3-1)}$$

$$+ \frac{s_1 n_1 s_2 n_2}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1 - 1, n_2 - 1, n_3) D^{(s_1-1)} D^{(s_2-1)} h^{(s_3)}$$

$$+ \frac{s_2 n_2 s_3 n_3}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1, n_2 - 1, n_3 - 1) h^{(s_1)} D^{(s_2-1)} D^{(s_3-1)}\Big]$$

$$\mathcal{L}_I^{(3,0)} = \sum\int \Big[+ \frac{s_3 n_3 s_2 n_2 s_1 n_1}{2} C_{n_1,n_2,n_3}^{s_1,s_2,s_3} \hat{T}(Q_{ij} \mid n_1 - 1, n_2 - 1, n_3 - 1) D^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)}\Big]$$

$$\mathcal{L}_I^{(0,1)} = \mathcal{L}_I^{(0,2)} = 0$$

In de Donder gauge
Trace decouples !!!

$$\mathcal{L}_I^{(0,3)} = \sum\int C^{s_1,s_2,s_3}_{n_1,n_2,n_3} \frac{Q_{12}Q_{23}Q_{31}}{8} [\hat{T}(Q_{12}-1, Q_{23}-1, Q_{31}-1 \,|\, n_1, n_2, n_3) \Box_a h^{(s_1)} \Box_b h^{(s_2)} \Box_c h^{(s_3)}]$$

$$\mathcal{L}_I^{(1,1)} = \sum\int C^{s_1,s_2,s_3}_{n_1,n_2,n_3} [+\frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12}-1, Q_{23}, Q_{31} \,|\, n_1, n_2-1, n_3) D^{(s_1-1)} \Box_b h^{(s_2)} h^{(s_3)}$$

$$+\frac{s_2 Q_{23} n_3}{4} \hat{T}(Q_{12}, Q_{23}-1, Q_{31} \,|\, n_1, n_2, n_3-1) h^{(s_1)} D^{(s_2-1)} \Box_c h^{(s_3)}$$

$$+\frac{s_3 Q_{31} n_1}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31}-1 \,|\, n_1-1, n_2, n_3) \Box_a h^{(s_1)} h^{(s_2)} D^{(s_3-1)}]$$

$$\mathcal{L}_I^{(1,2)} = \sum\int \; C^{s_1,s_2,s_3}_{n_1,n_2,n_3} [+\frac{s_1 Q_{12} Q_{23} n_3}{8} \hat{T}(Q_{12}-1, Q_{23}-1, Q_{31} \mid n_1, n_2, n_3-1) D^{(s_1-1)} \Box_b h^{(s_2)} \Box_c h^{(s_3)}$$

$$+\frac{s_2 Q_{23} Q_{31} n_1}{8} \hat{T}(Q_{12}, Q_{23}-1, Q_{31}-1 \mid n_1-1, n_2, n_3) \Box_a h^{(s_1)} D^{(s_2-1)} \Box_c h^{(s_3)}$$

$$+\frac{s_3 Q_{31} Q_{12} n_2}{8} \hat{T}(Q_{12}-1, Q_{23}, Q_{31}-1 \mid n_1, n_2-1, n_3) \Box_a h^{(s_1)} \Box_b h^{(s_2)} D^{(s_3-1)}]$$

$$\mathcal{L}_I^{(2,1)} = \sum\int \; C^{s_1,s_2,s_3}_{n_1,n_2,n_3} \; [+\frac{s_2 s_3 Q_{31} n_1 n_2}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31}-1 \mid n_1-1, n_2-1, n_3) \Box_a h^{(s_1)} D^{(s_2-1)} D^{(s_3-1)}$$

$$+\frac{s_1 s_2 Q_{23} n_3 n_1}{4} \hat{T}(Q_{12}, Q_{23}-1, Q_{31} \mid n_1-1, n_2, n_3-1) D^{(s_1-1)} D^{(s_2-1)} \Box_c h^{(s_3)}$$

$$+\frac{s_3 s_1 Q_{12} n_2 n_3}{4} \hat{T}(Q_{12}-1, Q_{23}, Q_{31} \mid n_1, n_2-1, n_3-1) D^{(s_1-1)} \Box_b h^{(s_2)} D^{(s_3-1)}]$$

D / Trh	0	1	2	3
0	hhh	Dhh	DDh	DDD
1	000	D Trh h	DD Trh	
2	000	D Trh Trh		
3	Trh TrhTrh			

4. Generating Function for HS cubic interactions

Sagnotti-Taronna GF (On-Shell)

$$\mathcal{A}^{00}(\Phi(z)) = \int_{z_1,z_2,z_3} \delta(z - z_{1,2,3}) exp\hat{W} \times \Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \mid_{a_1=a_2=a_3=0}$$

Where $\int_{z_1,z_2,z_3} \delta(z - z_{1,2,3}) = \int_{z_1,z_2,z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3)$

With vertex operator

$$\hat{W} = [(\partial_{a_1}\partial_{a_2}) + 1](\partial_{a_3}\nabla_{12}) + [(\partial_{a_2}\partial_{a_3}) + 1](\partial_{a_1}\nabla_{23}) + [(\partial_{a_3}\partial_{a_1}) + 1](\partial_{a_2}\nabla_{31})$$

This result is derived from String Theory side and in complete agreement with results presented here, derived by pure field theory approach!

Generalization to *Off-Shell Generating Function*

$$\eta_{a_1}, \bar{\eta}_{a_1}, \eta_{a_2}, \bar{\eta}_{a_2}, \eta_{a_3}, \bar{\eta}_{a_3}$$

Correspond to ghosts in String Theory?

Anticommuting variables!

$$\mathcal{A}(\Phi(z)) = \int d\eta_{a_1} d\bar{\eta}_{a_1} d\eta_{a_2} d\bar{\eta}_{a_2} d\eta_{a_3} d\bar{\eta}_{a_3} (1+\eta_{a_1}\bar{\eta}_{a_1})(1+\eta_{a_2}\bar{\eta}_{a_2})(1+\eta_{a_3}\bar{\eta}_{a_3})$$
$$exp\hat{W}\Phi(z_1;a_1)\Phi(z_2;a_2)\Phi(z_3;a_3)\,|_{a_1=a_2=a_3=0}$$

Where

$$\begin{aligned}\hat{W} = {} & [1+(\partial_{a_1}\partial_{a_2}) + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_1}\Box_{a_1}][(\partial_{a_3}\nabla_{12}) + \eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \eta_{a_3}\bar{\eta}_{a_2}D_{a_2}] \\ & +[1+(\partial_{a_2}\partial_{a_3}) + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_2}\Box_{a_2}][(\partial_{a_1}\nabla_{23}) + \eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \eta_{a_1}\bar{\eta}_{a_3}D_{a_3}] \\ & +[1+(\partial_{a_3}\partial_{a_1}) + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_3}\Box_{a_3}][(\partial_{a_2}\nabla_{31}) + \eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \eta_{a_2}\bar{\eta}_{a_1}D_{a_1}]\end{aligned}$$

*Off-shell*ing the *On-shell expressions*

$$(\partial_{a_1}\partial_{a_2}) \rightarrow (\partial_{a_1}\partial_{a_2}) + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_1}\Box_{a_1},$$

$$(\partial_{a_2}\partial_{a_3}) \rightarrow (\partial_{a_2}\partial_{a_3}) + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_2}\Box_{a_2},$$

$$(\partial_{a_3}\partial_{a_1}) \rightarrow (\partial_{a_3}\partial_{a_1}) + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_3}\Box_{a_3},$$

$$(\partial_{a_1}\nabla_{23}) \rightarrow (\partial_{a_1}\nabla_{23}) + \eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \eta_{a_1}\bar{\eta}_{a_3}D_{a_3}$$

$$(\partial_{a_2}\nabla_{31}) \rightarrow (\partial_{a_2}\nabla_{31}) + \eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \eta_{a_2}\bar{\eta}_{a_1}D_{a_1}$$

$$(\partial_{a_3}\nabla_{12}) \rightarrow (\partial_{a_3}\nabla_{12}) + \eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \eta_{a_3}\bar{\eta}_{a_2}D_{a_2}.$$

Classification of Lagrangian terms

$\square_{a_i}$ \ D_{a_i}	0	1	2	3
0	$\mathcal{A}^{00}$	0	$\mathcal{A}^{20}$	0
1	0	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	$\mathcal{A}^{02}$	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

Intriguing case

$$W = (\partial_{a_1}\partial_{a_2})(\partial_{a_3}\nabla_{12}) + (\partial_{a_2}\partial_{a_3})(\partial_{a_1}\nabla_{23}) + (\partial_{a_3}\partial_{a_1})(\partial_{a_2}\nabla_{31})$$

The Lagrangian generated by this operator does not mix different HS gauge fields in the interaction! This includes *only minimal selfinteractions* for even spin gauge fields which is a *closed subset of all cubic vertices*. This gives hope for nonlinear selfinteraction theory of even HS gauge fields like Einstein-Hilbert gravity.

Simple example: Cubic selfinteraction of the graviton in deDonder gauge

$$L_1(h^{(2)}) = \frac{1}{2} h^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{\mu\nu}h^{\mu\nu} + h^{\alpha\mu}\nabla_\alpha h^{\beta\nu}\nabla_\beta h_{\mu\nu}$$

5. Conclusions

- It was shown that there is a local, higher derivative Lagrangian for HS interactions in flat space-time, at least for first nontrival order – cubic couplings.
- All possible cases of cubic interactions between different HS gauge fields including selfinteraction of even spin fields are presented in one compact formula.
- These interactions between HS gauge fields are unique, therefore reproduce *flat limit of the Fradkin-Vasiliev vertex for cubic couplings of HS gauge fields to the linearized gravity*, and also include all lower spin cases which are well known for many years.
- It was shown that these cubic interactions are in exact agreement with the ones obtained from String Theory in tensionless limit.
- The simple and deep structure of these interactions give hope that they can be continued to the quartic and higher order Lagrangians, which would lead to covariant Lagrangian formulation of full nonlinear HS gauge field theory.

Thank You for your attention!