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N=4 Multi-Particle Mechanics, WDVV Equations and Deformed Root Systems

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- Conformal quantum mechanics: Calogero system
- $N{=}4$ superconformal extension: $su(1,1|2)$ algebra
- The structure equations for (F,U): WDVV, flatness, homogeneity
- <u>first $U{=}0$</u>: covector ansatz for prepotential F
- Partial isometry formulation ($\Leftrightarrow$ WDVV, Veselov system)
- Deformed root systems and polytopes
- Hypergraphs and matroids
- Superspace approach: inertial coordinates in $\mathbb{R}^{n+1}$
- <u>now $U{\neq}0$</u>: three- and four-particle solutions
- Summary

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Conformal quantum mechanics: Calogero system

$n{+}1$ identical particles with unit mass, moving on the real line

Hamiltonian: $$H \;=\; \tfrac{1}{2}p_i p_i \;+\; V_B(x^1,\ldots,x^{n+1})$$

$so(1,2)$: $$[D,H] \;=\; -\mathrm{i}H\,, \quad [H,K] \;=\; 2\mathrm{i}D\,, \quad [D,K] \;=\; \mathrm{i}K$$

quantization: $$[x^i,p_j] \;=\; \mathrm{i}\delta_j{}^i$$

realization: $$D \;=\; -\tfrac{1}{4}(x^i p_i + p_i x^i) \qquad \text{and} \qquad K \;=\; \tfrac{1}{2}x^i x^i$$

conformal invariance $\longrightarrow$ $$(x^i\partial_i + 2)\,V_B \;=\; 0$$

demand also permutation and translation invariance and two-body forces only

$$\longrightarrow \qquad V_B \;=\; \sum_{i<j}\frac{g^2}{(x^i-x^j)^2}$$ Calogero model

$\mathcal{N}$=4 superconformal extension: $su(1,1|2)$ algebra

extend $so(1,2)$ to $su(1,1|2)$: $\quad (H,D,K) \rightarrow (H,D,K,Q_\alpha,S_\alpha,J_a,C)$

with $\alpha=1,2$, $a=1,2,3$ and $(Q_\alpha)^\dagger{=}\bar{Q}^\alpha$, $(S_\alpha)^\dagger{=}\bar{S}^\alpha$ and central charge C

nonvanishing (anti)commutators:

$$[D,H] = -\,\mathrm{i}H \qquad [H,K] = 2\mathrm{i}D$$

$$[D,K] = +\mathrm{i}K \qquad [J_a,J_b] = \mathrm{i}\epsilon_{abc}J_c$$

$$\{Q_\alpha,\bar{Q}^\beta\} = 2H\delta_\alpha{}^\beta \qquad \{Q_\alpha,\bar{S}^\beta\} = +2\mathrm{i}(\sigma_a)_\alpha{}^\beta J_a - 2D\delta_\alpha{}^\beta - \,\mathrm{i}C\delta_\alpha{}^\beta$$

$$\{S_\alpha,\bar{S}^\beta\} \;= 2K\delta_\alpha{}^\beta \qquad \{\bar{Q}^\alpha,S_\beta\} = -\,2\mathrm{i}(\sigma_a)_\beta{}^\alpha J_a - 2D\delta_\beta{}^\alpha + \mathrm{i}C\delta_\beta{}^\alpha$$

$$[D,Q_\alpha] = -\tfrac{\mathrm{i}}{2}Q_\alpha \qquad [D,S_\alpha] = +\tfrac{\mathrm{i}}{2}S_\alpha$$

$$[K,Q_\alpha] = +\mathrm{i}S_\alpha \qquad [H,S_\alpha] = -\mathrm{i}Q_\alpha$$

$$[J_a,Q_\alpha] = -\tfrac{1}{2}(\sigma_a)_\alpha{}^\beta Q_\beta \qquad [J_a,S_\alpha] = -\tfrac{1}{2}(\sigma_a)_\alpha{}^\beta S_\beta$$

$$[D,\bar{Q}^\alpha] = -\tfrac{\mathrm{i}}{2}\bar{Q}^\alpha \qquad [D,\bar{S}^\alpha] = +\tfrac{\mathrm{i}}{2}\bar{S}^\alpha$$

$$[K,\bar{Q}^\alpha] = +\mathrm{i}\bar{S}^\alpha \qquad [H,\bar{S}^\alpha] = -\mathrm{i}\bar{Q}^\alpha$$

$$[J_a,\bar{Q}^\alpha] = \tfrac{1}{2}\bar{Q}^\beta(\sigma_a)_\beta{}^\alpha \qquad [J_a,\bar{S}^\alpha] = \tfrac{1}{2}\bar{S}^\beta(\sigma_a)_\beta{}^\alpha$$

fermionic variables: $\psi^i_\alpha,\ \bar\psi^{i\alpha}={\psi^i_\alpha}^\dagger$ with $i=1,\dots,n{+}1$ and $\alpha=1,2$

$$\{\psi^i_\alpha,\psi^j_\beta\}=0\,,\qquad \{\bar\psi^{i\alpha},\bar\psi^{j\beta}\}=0\,,\qquad \{\psi^i_\alpha,\bar\psi^{j\beta}\}\ =\ \delta_\alpha{}^\beta\delta^{ij}$$

$$Q_{0\alpha}\ =\ p_i\psi^i_\alpha\,,\qquad \bar Q_0^\alpha\ =\ p_i\bar\psi^{i\alpha}\qquad \text{and}\qquad S_{0\alpha}\ =\ x^i\psi^i_\alpha\,,\qquad \bar S_0^\alpha\ =\ x^i\bar\psi^{i\alpha}$$

$$H_0=\tfrac12 p_ip_i\,,\quad D_0=-\tfrac14(x^ip_i+p_ix^i)\,,\quad K_0=\tfrac12 x^ix^i\,,\quad J_{0a}=\tfrac12\bar\psi^{i\alpha}(\sigma_a)_\alpha{}^\beta\psi^i_\beta$$

free generators fail to obey $su(1,1|2)$ algebra $\longrightarrow$ interactions are needed!

$$Q_\alpha\ =\ Q_{0\alpha}-\mathrm{i}\,[S_{0\alpha},V]\qquad \text{and}\qquad H\ =\ H_0+V$$

minimal ansatz to close the algebra [Wyl99,BGL04]: Weyl ordering $\langle\dots\rangle$

$$\boxed{V\ =\ V_B(x)\ -\ U_{ij}(x)\langle\psi^i_\alpha\bar\psi^{j\alpha}\rangle\ +\ \tfrac14 F_{ijkl}(x)\langle\psi^i_\alpha\psi^{j\alpha}\bar\psi^{k\beta}\bar\psi^l_\beta\rangle}$$

U_{ij} and F_{ijkl} are totally symmetric & homogeneous of degree -2 in $\{x^1,\dots,x^n\}$

$$\longrightarrow\qquad Q_\alpha\ =\ \big(p_j-\mathrm{i}\,x^i\,U_{ij}(x)\big)\,\psi^j_\alpha\ -\tfrac{\mathrm{i}}{2}\,x^i\,F_{ijkl}(x)\,\langle\psi^j_\beta\,\psi^{k\beta}\bar\psi^l_\alpha\rangle$$

$\boxed{\text{The structure equations for } (F,U)\text{: WDVV, flatness, homogeneity}}$

insert minimal V ansatz into $su(1,1|2)$ algebra and demand closure $\longrightarrow$

$$U_{ij} \;=\; \partial_i\partial_j U \qquad \text{and} \qquad F_{ijkl} \;=\; \partial_i\partial_j\partial_k\partial_l F$$

two scalar prepotentials U and F, subject to "structure equations" [Wyl99,BGL04]

$$\boxed{(\partial_i\partial_k\partial_p F)(\partial_p\partial_l\partial_j F) \;=\; (\partial_i\partial_l\partial_p F)(\partial_p\partial_k\partial_j F)} \quad , \qquad x^i\partial_i\partial_j\partial_k F \;=\; -\delta_{jk}$$

$$\boxed{\partial_i\partial_j U - (\partial_i\partial_j\partial_k F)\,\partial_k U \;=\; 0} \quad , \qquad x^i\partial_i U \;=\; -C$$

linear in U (flatness), quadratic in F (WDVV), with homogeneity properties

redundancy: $U \simeq U + \text{constant}\,, \quad F \simeq F + \text{quadratic polynomial}$

potential: $V_B \;=\; \frac{1}{2}(\partial_i U)(\partial_i U) \;+\; \frac{\hbar^2}{8}(\partial_i\partial_j\partial_k F)(\partial_i\partial_j\partial_k F)$

consequence: $x^i\,F_{ijkl} \;=\; -\partial_j\partial_k\partial_l F \qquad \text{and} \qquad x^i\,U_{ij} \;=\; -\partial_j U$

first $U{=}0$: covector ansatz for prepotential F

$$x^i \partial_i \partial_j \partial_k F = -\delta_{jk} \quad \Rightarrow \quad (x^i \partial_i - 2) F = -\tfrac{1}{2}\, x^i x^i \quad \Rightarrow \quad F \sim x^2 \ln|x| + F_{\text{hom}}$$

breaks translation invariance! homogeneity conditions solved by ansatz [Wyl99]

$$\boxed{F = -\tfrac{1}{2} \sum_{\alpha} (\alpha{\cdot}x)^2 \ln|\alpha{\cdot}x|} \qquad \text{with} \qquad \boxed{\sum_{\alpha} \alpha_i \alpha_j = \delta_{ij}}$$

covectors $\quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \quad \in (\mathbb{R}^{n+1})^* \quad$ or $\quad \in \mathrm{i}(\mathbb{R}^{n+1})^*$

in $\quad \alpha(x) = \alpha{\cdot}x = \alpha_i x^i \qquad\qquad \#\alpha =: p$

coupling constants g and central charge C reside in prepotential U

remaining structure equations take the form

“WDVV” $$\boxed{\sum_{\alpha,\beta} \frac{\alpha\cdot\beta}{\alpha\cdot x\ \beta\cdot x}\,(\alpha_i\beta_j-\alpha_j\beta_i)(\alpha_k\beta_l-\alpha_l\beta_k)\ =\ 0}$$ [MarGra99,Ves99]

“flatness” $$\boxed{\partial_i\partial_j U\ +\ \sum_\alpha \alpha_i\alpha_j\,\frac{\alpha\cdot\partial U}{\alpha\cdot x}\ =\ 0}$$

potential: $$V_B\ =\ \tfrac12(\partial U)\cdot(\partial U)\ +\ \tfrac{\hbar^2}{8}\sum_{\alpha,\beta}\frac{(\alpha\cdot\beta)^3}{\alpha\cdot x\ \beta\cdot x}$$

special solutions ($C{=}0$): $U\equiv 0 \quad\Rightarrow\quad V_B = O(\hbar^2)$

strategy: first solve WDVV, then turn on flat U in this F background

WDVV homogeneous $\Rightarrow$ covectors projective, norm via $\boxed{\sum_\alpha \alpha\otimes\alpha\ =\ \mathbb{1}}$

it suffices to consider indecomposable covector sets $\{\alpha\}$

decouple center of mass $\alpha_{\text{com}} = (1,1,\ldots,1)$, reducing $(\mathbb{R}^{n+1})^* \to (\mathbb{R}^n)^*$

partial results known for $n{\leq}3$ [Wyl99,ChaVes01,BGL04,GLP07,FeiVes07,GLP08]

Partial isometry formulation

$\sum_\alpha \alpha\otimes\alpha \propto \mathbb{1} \quad \Rightarrow \quad \text{WDVV}\cdot x = 0 \quad \Rightarrow \quad \frac{1}{12}n(n-1)^2(n-2)$ indep't equations

$$\sum_{\alpha,\beta} \frac{\alpha\cdot\beta}{\alpha\cdot x\;\beta\cdot x}\,(\alpha\wedge\beta)^{\otimes 2} \;=\; 0 \qquad \Longrightarrow \qquad \sum_{\substack{\alpha,\beta\\(\alpha\neq\beta)}} \frac{\alpha\cdot\beta}{\alpha\cdot x\;\beta\cdot x} \;=\; 0 \quad \forall\,\pi$$

decompose into separate equations for each plane $\pi\in\Lambda^2((\mathbb{R}^n)^*)$. Three cases:

case a) π contains zero or one covector $\Rightarrow$ trivial

case b) π contains two covectors, $\pi=\alpha\wedge\beta$ $\Rightarrow$ orthogonality $\alpha\cdot\beta=0$

case c) π contains $q(>2)$ covectors $\Rightarrow$ nontrivial condition on π:

$$(*) \quad \boxed{\sum_{\alpha\in\pi} \alpha\otimes\alpha \;=\; \lambda_\pi\,\mathbb{1}_\pi \;=:\; \lambda_\pi\,P_\pi} \qquad \text{for} \quad \lambda_\pi\in\mathbb{R} \quad \text{and} \quad \text{rank}(P_\pi)=2$$

yields WDVV equation on π which is trivially fulfilled for $n{=}2$ $\quad\checkmark$

reformulate condition $(*)$ in terms of partial isometries:

$n{\times}p$ matrix $A = \big(\alpha_{ia}\big)_{a=1,\dots,p}^{i=1,\dots,n}$ defines a map $A : \mathbb{R}^p \to \mathbb{R}^n$ with $A\,A^\top = \mathbb{1}_n$

for each nontrivial plane π, select all $\alpha \in \pi$ via $B_\pi : \mathbb{R}^p \to \mathbb{R}^q$ by $\{\alpha\} \mapsto \{\alpha_s\}$

so that $A_\pi \ := \ A\,B_\pi^\top \ = \ \big(\alpha_{ia_s}\big)_{s=1,\dots,q}^{i=1,\dots,n}$ maps $R^q \to \mathbb{R}^n$

condition $(*)$ means $\boxed{A_\pi\,A_\pi^\top \ = \ \lambda_\pi\,P_\pi \qquad \Leftrightarrow \qquad A_\pi^\top A_\pi \ = \ \lambda_\pi\,Q_\pi}$ $(**)$

with projectors P_π on $\mathbb{R}^n$ and Q_π on $\mathbb{R}^q$ of rank two and multipliers λ_π. Thus:

A is a solution iff $\frac{A_\pi}{\sqrt{\lambda_\pi}}$ is a rank-2 partial isometry $(**)$ for each nontrivial plane π

$\Rightarrow \quad A_\pi\,A_\pi^\top A_\pi \ = \ \lambda_\pi\,A_\pi$ note: $A \neq A_\pi\,B_\pi$ A_π splits over $\mathbb{R}^2$

$\Rightarrow \quad \exists \quad D_\pi : \mathbb{R}^q \to \mathbb{R}^2$ and $C_\pi : \mathbb{R}^2 \leftarrow \mathbb{R}^n$ such that $A_\pi \ = C_\pi^\top D_\pi$

$$
\begin{tikzcd}
& \mathbb{R}^p \arrow[dl, "B_\pi"'] \arrow[dr, Rightarrow, "A"] & \\
\mathbb{R}^q \arrow[rr, Rightarrow, "A_\pi"] \arrow[dr, "D_\pi"'] \arrow[loop left, "Q_\pi"] & & \mathbb{R}^n \arrow[dl, "C_\pi"] \arrow[loop right, "P_\pi"] \\
& \mathbb{R}^2 &
\end{tikzcd}
$$

example $n{=}3$, $p{=}6$:

$$
A = \frac{1}{6}\;
\begin{array}{cccccc}
\alpha & \beta & \gamma & \alpha' & \beta' & \gamma' \\
\end{array}
$$

$$
A = \frac{1}{6}\begin{pmatrix} 6t & -3t & -3t & 0 & 3w & -3w \\ 0 & 3\sqrt{3}t & -3\sqrt{3}t & -2\sqrt{3}w & \sqrt{3}w & \sqrt{3}w \\ 0 & 0 & 0 & 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix}
$$

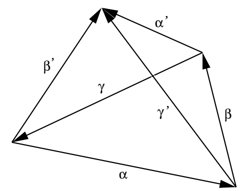
$$t \in [-\sqrt{\tfrac{2}{3}}, +\sqrt{\tfrac{2}{3}}\,]$$

$$w = \sqrt{2 - 3t^2}$$

$$A\,A^\top = \mathbb{1}$$

nontrivial planes: $\langle\alpha\,\beta\,\gamma\rangle$, $\langle\alpha\,\beta'\gamma'\rangle$, $\langle\alpha'\beta\,\gamma'\rangle$, $\langle\alpha'\beta'\gamma\rangle$

orthogonality: $\alpha\cdot\alpha' = \beta\cdot\beta' = \gamma\cdot\gamma' = 0$



$$
A_{\langle\alpha\,\beta\,\gamma\rangle} = \frac{1}{2}\begin{pmatrix} 2t & -t & -t \\ 0 & \sqrt{3}t & -\sqrt{3}t \\ 0 & 0 & 0 \end{pmatrix} \quad\Rightarrow\quad A_\pi\,A_\pi^\top = \tfrac{3}{2}t^2\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tfrac{3}{2}t^2\cdot P_\pi
$$

$$
A_{\langle\alpha\,\beta'\gamma'\rangle} = \frac{1}{6}\begin{pmatrix} 6t & 3w & -3w \\ 0 & \sqrt{3}w & \sqrt{3}w \\ 0 & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix} \quad\Rightarrow\quad A_\pi\,A_\pi^\top = \frac{1-\frac{1}{2}t^2}{6-3t^2}\begin{pmatrix} 6-3t^2 & 0 & 0 \\ 0 & 2-3t^2 & 2w \\ 0 & 2w & 4 \end{pmatrix}
$$

Deformed root systems and polytopes

take $\alpha \in \Phi^+ \;=\;$ set of positive roots of a simple Lie algebra [MarGra99,Ves99]

$\Phi^+ = \Phi^+_{\mathsf{L}} \cup \Phi^+_{\mathsf{S}}$ long & short roots canonical normalization $\alpha{\cdot}\alpha = \frac{2}{r}$

need scale factors $\{f_\alpha\} = \{f_{\mathsf{L}}, f_{\mathsf{S}}\}$ in F to satisfy $\sum_\alpha f_\alpha\, \alpha \otimes \alpha \;=\; \mathbb{1}$

prepotential

$$\boxed{F(t) \;=\; -\tfrac{1}{2}\Big(f_{\mathsf{L}}(t) \sum_{\alpha\in\Phi^+_{\mathsf{L}}} + f_{\mathsf{S}}(t) \sum_{\alpha\in\Phi^+_{\mathsf{S}}}\Big)\, (\alpha{\cdot}x)^2 \ln|\alpha{\cdot}x|}$$

with $f_{\mathsf{L}} \;=\; \frac{1}{h^\vee} + (h - h^\vee)\, t$ and $f_{\mathsf{S}} \;=\; \frac{1}{h^\vee} + (h - r h^\vee)\, t$

normalization: $f_{\mathsf{L}} \sum_{\alpha\in\Phi^+_{\mathsf{L}}} \alpha \otimes \alpha \;+\; f_{\mathsf{S}} \sum_{\alpha\in\Phi^+_{\mathsf{S}}} \alpha \otimes \alpha \;=\; \mathbb{1}$ partition of unity

WDVV ✓ due to $\displaystyle\sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \frac{\alpha{\cdot}\beta}{\alpha{\cdot}x\;\beta{\cdot}x} \;=\; 0$ for $\alpha, \beta \;\in\; \Phi^+ \cap \Pi$ $\forall \Pi$

one-parameter family! $t{=}0$ solutions were known [MarGra99,Ves99] example $A_n \oplus A_1$: $\{\alpha\} = \{\, e_i - e_j\,,\ \sum_i e_i \ \mid \ 1 \le i < j \le n+1\,\}$

$$F_{A_n \oplus A_1} = -\frac{1/2}{n+1} \sum_{i<j} (x^i - x^j)^2 \ln|x^i - x^j| - \frac{1/2}{n+1} (\textstyle\sum_i x^i)^2 \ln|\textstyle\sum_i x^i|$$

example G_2: $\{\alpha\} = \left\{ \frac{1}{\sqrt{3}}(e_i - e_j)\,,\ \ \frac{1}{\sqrt{3}}(e_i + e_j - 2e_k) \ \middle|\ (i,j,k) \ \text{ cyclic} \right\}$

$$F_{G_2} = -\tfrac{1}{6} f_S\, (x^1 - x^2)^2 \ln|x^1 - x^2| - \tfrac{1}{6} f_L\, (x^1 + x^2 - 2x^3)^2 \ln|2x^1 - x^2 - x^3| + \text{cyclic}$$

with $p = 6$ and $f_S = \frac{1}{4} - 6\,t$, $f_L = \frac{1}{4} + 2\,t$

note: center-of-mass decoupling $\Leftrightarrow$ projection $\perp$ $e_1 + e_2 + e_3$

can we deform the root system solutions (change angles between covectors)?

example of A_n: $p = \frac{1}{2}n(n+1)$ $\sum_\alpha f_\alpha \, \alpha \otimes \alpha = \mathbb{1}$ uniquely fixes $\{f_\alpha\}$

ortho-polytope idea: $\{\alpha\}$ form the edges of a suitable n-simplex
incidence of $\triangle$s take care of WDVV, non-concurrent edges must be orthogonal

counting:

	ray moduli	incidences	simplex moduli	orthogonality	final moduli
#	$\frac{1}{2}n^2(n-1)$	$-\frac{1}{2}(n-2)(n^2-1)$	$\frac{1}{2}(n-1)(n+2)$	$-\frac{1}{2}(n-2)(n+1)$	n
$n=2,3,4$	2, 9, 24	0, −4, −15	2, 5, 9	0, −2, −5	2, 3, 4

final moduli space $\mathcal{M}(A_n)$ is that of orthocentric n-simplices cf. [ChaVes01]

previous example was 1-parameter family in $\mathcal{M}(A_3)$ other weight systems?

ex.: B_3 roots $\underline{21}$ $\overset{\times 4}{\Longrightarrow}$ edge set of truncated cube $\Longrightarrow$ deform: cuboid

$$\{\alpha{\cdot}x\} = \{d_1x^1,\ d_2x^2,\ d_3x^3\,;\ c_3(c_2x^1 \pm c_1x^2)\,,\ c_1(c_3x^2 \pm c_2x^3)\,,\ c_2(c_1x^3 \pm c_3x^1)\}$$

with $\{f_\alpha\} = \left\{\frac{c_0^2+c_1^2-c_2^2-c_3^2}{c^2\,d_1^2}\,,\ \frac{c_0^2-c_1^2+c_2^2-c_3^2}{c^2\,d_1^2}\,,\ \frac{c_0^2-c_1^2-c_2^2+c_3^2}{c^2\,d_1^2}\,;\ \frac{1}{c^2\,c_3^2}\,,\ \frac{1}{c^2\,c_1^2}\,,\ \frac{1}{c^2\,c_2^2}\right\}$

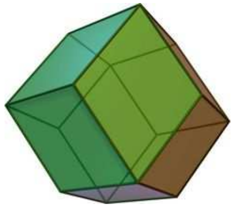
and $c^2 := c_0^2+c_1^2+c_2^2+c_3^2$ with $c_0, c_i, d_i \in \mathbb{R}$;

combinations $\sqrt{f_\alpha}\,\alpha$ depend only on $\frac{c_i}{c_0}$ $\longrightarrow$ 3 parameters

ex.: B_3 weights $\underline{7} \oplus \underline{8}$ $\overset{\times(4,6)}{\Longrightarrow}$ edges of rhombic dodecahedron $\Longrightarrow$ deform:

$$\alpha{\cdot}x = d_1x^1,\ \beta{\cdot}x = d_2x^2,\ \gamma{\cdot}x = d_3x^3;\ \frac{\alpha+\beta+\gamma}{2},\ \frac{\alpha-\beta-\gamma}{2},\ \frac{-\alpha+\beta-\gamma}{2},\ \frac{-\alpha-\beta+\gamma}{2}$$

with $f_\alpha = \frac{-d_1^2+d_2^2+d_3^2}{d^2\,d_1^2}\,,\quad f_\beta = \frac{d_1^2-d_2^2+d_3^2}{d^2\,d_2^2}\,,\quad f_\gamma = \frac{d_1^2+d_2^2-d_3^2}{d^2\,d_3^2}\,;\quad f_{\text{spinor}} = \frac{2}{d^2}$



and $d^2 := d_1^2+d_2^2+d_3^2$; faces dissected into triangles ;

combinations $\sqrt{f_\alpha}\,\alpha$ depend only on $\frac{d_i}{d_j}$ $\longrightarrow$ 2 parameters

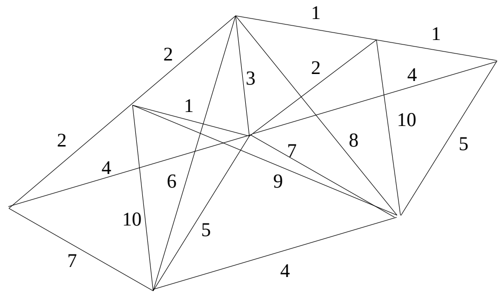
geometry: WDVV fulfilled due to the edge incidence relations of the polytopes

Hypergraphs and matroids

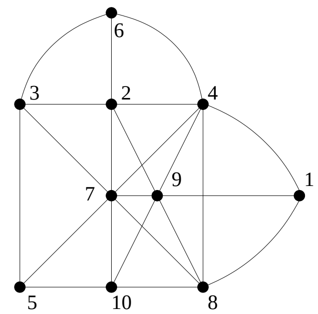
ortho-polytopes fail: construction not unique, edge multiplicities, internal ‘edges’

first counterexample at $n{=}3$, $p{=}10$: $A =$

$$\frac{1}{2}\begin{pmatrix} 2\sqrt{6} & 2\sqrt{6} & 4 & 0 & 2 & -2 & 2\sqrt{3} & -2\sqrt{3} & 0 & 0 \\ 2\sqrt{6} & -2\sqrt{6} & 0 & 0 & 3 & 3 & -\sqrt{3} & -\sqrt{3} & -3\sqrt{2} & \sqrt{6} \\ 0 & 0 & 0 & 4\sqrt{3} & 1 & 1 & \sqrt{3} & \sqrt{3} & \sqrt{2} & \sqrt{6} \end{pmatrix}$$



better abstract to the incidence structure / set system / hypergraph [Wikipedia]:
a generalization of a graph, where an edge can connect any number of vertices



here: covector $\longrightarrow$ vertex and plane $\longrightarrow$ hyperedge

for simplicity drop all 2-vertex hyperedges

simple: each hyperedge is maximal ($\supset$ no smaller hyperedge)
linear: the intersection of two hyperedges has at most one vertex
complete: each vertex pair is contained in some hyperedge
irreducible: the hypergraph (without 2-vertex hyperedges) is connected
orthogonal: each vertex pair in a (dropped) 2-vertex hyperedge is ‘orthogonal’

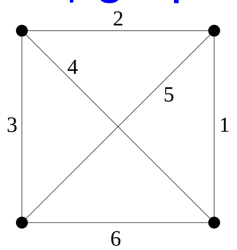
program: construct all linear simple complete irreducible orthogonal hypergraphs for given (n,p) and check the partial isometry conditions $(**)$ for all planes π

problem: orthogonality is not a natural hypergraph property but depends on the dimension n of a possible covector realization almost nothing known

better: capture abstractly the essence of linear dependence $\longrightarrow$ matroids

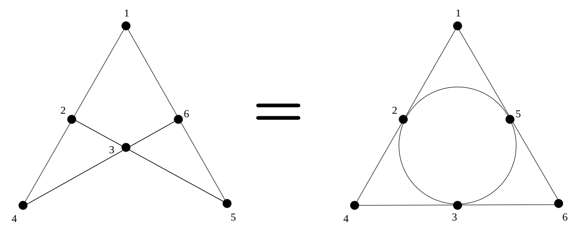
best illustrated with an example of a graphical matroid:

K_4 graph $\rightarrow$ set of minimal circuits = matroid $\rightarrow$ geometric representation



$$\{\{124\},\{156\},\{235\},\{346\},$$
$$\{1236\},\{1345\},\{2456\}\}$$

second row is irrelevant here



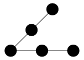
disadvantages: also capture higher-rank dependencies (only coplanarity needed); not always representable $\rightarrow$ need simple irred. orthogonal $\mathbb{R}$-vector matroids

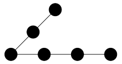
advantages: contraction (and sometimes deletion) preserve WDVV; representability and orthogonality natural; geometric representation is in $\mathbb{R}^{n-1}$

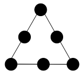
for n=3: simple matroids $\Leftrightarrow$ simple linear complete hypergraphs

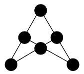
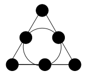
growth of rank n=3 matroids of cardinality p:

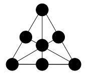
p	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
simple	0	1	2	4	9	23	68	383	5249	232928	28872972	$\cdots$
simple+irred.	0	0	0	1	3	12	41	307	4844	227614	28639650	$\cdots$
simple+irred. $\mathbb{R}$-vec.+ortho.	0	0	0	0	1	1	1	1	3	?	?	$\cdots$

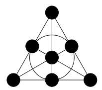
$\{\{123\},\{145\}\}$  $A_3\backslash\{6\}$ $\mathbb{R}$-vector but not orthogonal

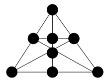
$\{\{123\},\{1456\}\}$  $B_3\backslash\{4,5,9\}$ $\mathbb{R}$-vector but not orthogonal

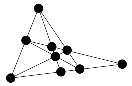
$\{\{123\},\{145\},\{356\}\}$  $D(2,1;\alpha)\backslash\{7\}$ not $\mathbb{R}$-vector

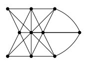
$\{\{123\},\{145\},\{356\},\{246\}\}$  =  $A_3(s,t,u)$

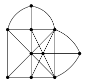
$\{\{123\},\{145\},\{356\},\{347\},\{257\},\{167\}\}$  $\underline{7}\oplus\underline{8}$ of B_3 $=$ $D(2,1;\alpha)(s,t)$

$\{\{123\},\{145\},\{356\},\{347\},\{257\},\{167\},\{246\}\}$  Fano matroid – not $\mathbb{R}$-vector

$\{\{123\},\{145\},\{356\},\{347\},\{257\},\{248\},\{1678\}\}$  $B_3\backslash\{9\}(s,t)$

$\{\{123\},\{145\},\{347\},\{257\},\{2489\},\{1678\},\{3569\}\}$  $B_3(s,t,u)$

$\{\{150\},\{167\},\{259\},\{268\},\{456\},\{479\},\{480\},\{1234\},\{3578\},\{3690\}\}$  $\subset AB(1,3)(t)$

$\{\{179\},\{289\},\{356\},\{378\},\{457\},\{468\},\{490\},\{1234\},\{1580\},\{2670\}\}$  $\subset AB(1,3)(t)$

we generate matroids or hypergraphs with a computer algebra program

have implemented simplicity, linearity, completeness, irreducibility, orthogonality

the program then generates a parametric covector realization if possible

on this we can finally test the partial-isometry conditions $(**)$

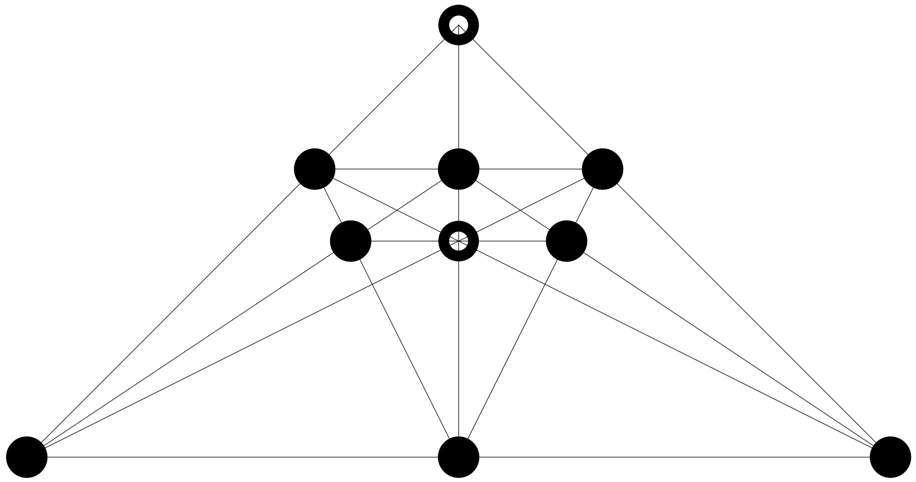
conjecture: our hypergraph/matroid subclass already $(**)$ $\Leftrightarrow$ WDVV solved

running the program . . .

check for $p \leq 10$ reveals

first counterexample at $p = 10$:

↯ ↯ ↯ ↯ ↯



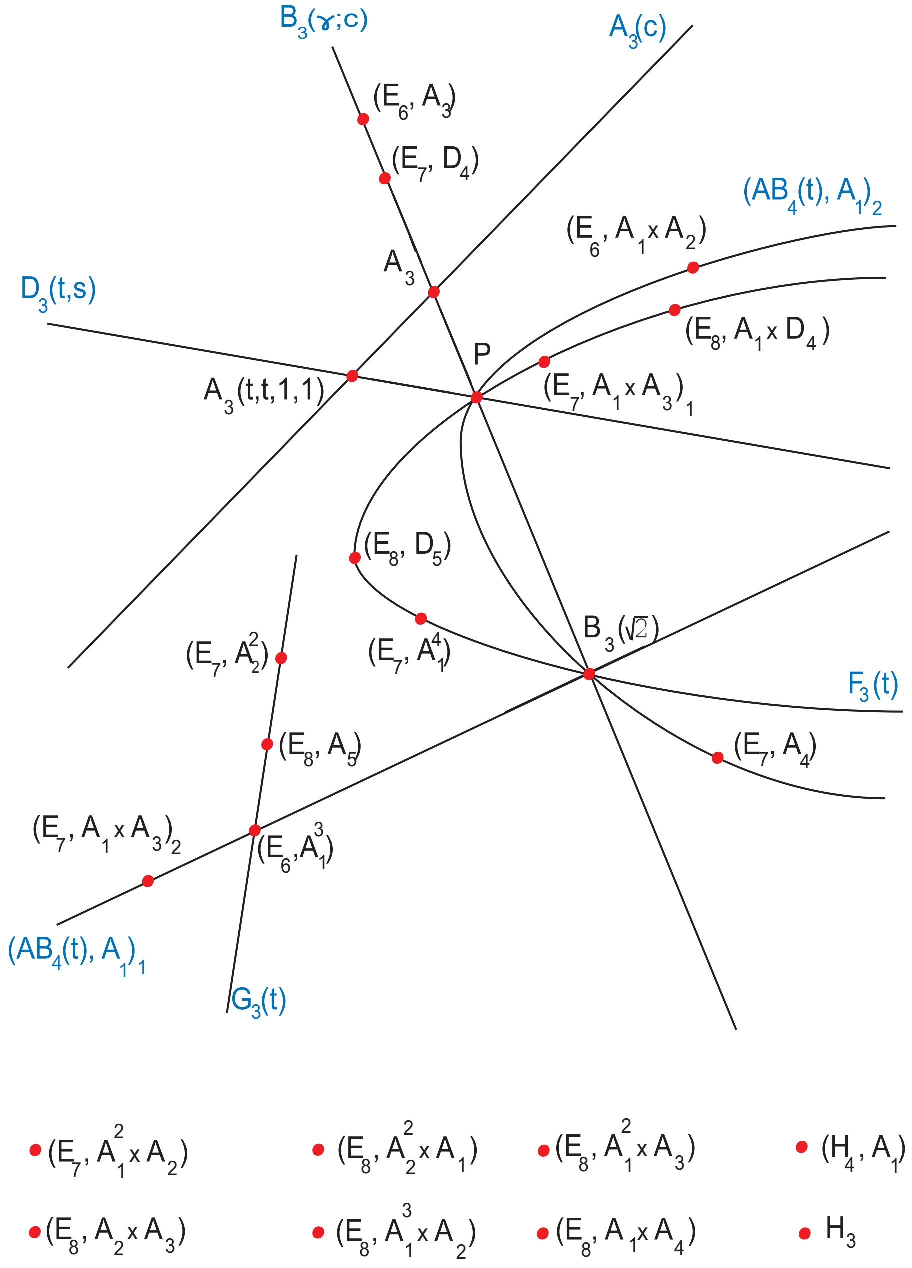


Figure 1: All known ∨-systems in dimension 3.

Superspace approach: inertial coordinates in $\mathbb{R}^{n+1}$

$\mathcal{N}$=4 superfields $\mathbb{u}^A(t,\theta^a,\bar\theta_a)$, with constraints $D^2\mathbb{u}^A = 0 = \overline{D}^2\mathbb{u}^A$

consequence: $[D^a,\overline{D}_a]\mathbb{u}^A = 4\,g^A$ with constants g^A and $A=1,\ldots,n+1$

$\mathcal{N}$=4 superconformal action for $\mathbb{u}^A(t,\theta,\bar\theta) = u^A(t) + O(\theta,\bar\theta)$:

$$S \;=\; -\int \mathrm{d}t\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,G(\mathbb{u}) \;=\; \tfrac12\int \mathrm{d}t\,\left[G_{AB}\,\dot u^A\dot u^B - 4G_{AB}\,g^A g^B + \text{fermions}\right]$$

with a superpotential $G(u)$ subject to $G - G_A u^A = \tfrac12 c_A u^A$ for constants c_A

want flat bose metric $\Leftrightarrow$ Riemann$(G_{AB}) = 0$ $\Leftrightarrow$ $\boxed{G_{A[BX}G^{XY}G_{YC]D} = 0}$

consequence: $\exists$ inertial coordinates x^i s.t. $S = \int \mathrm{d}t\,\left[\tfrac12\delta_{ij}\dot x^i\dot x^j - V_B^{\text{cl}}(x) + \ldots\right]$

goal: find admissible functions $u^A = u^A(x)$ and compute corresponding G, V_B^{cl}

integrability conditions $\boxed{\dfrac{\partial x^i}{\partial u^A}\big(u(x)\big) \equiv \Big(\big(u^{\bullet}_{\bullet}\big)^{-1}\Big)^i{}_A =: w_{A,i} \overset{!}{=} \partial_i w_A \equiv \dfrac{\partial w_A}{\partial x^i}(x)}$

consequences: $\exists\, F(x)$ s.t. $f_{ijk} := -w_{A,i}\, u^A_{jk} = \partial_i\partial_j\partial_k F$ obeys WDVV!

superconformality $x^i\, u^A_i = 2\, u^A \quad \Leftrightarrow \quad u^A$ are homogeneous quadratic in x

superpotential $G_{ij} + f_{ijk} G_k = -\delta_{ij}$ and $x^i G_i - 2G + \frac{1}{2} x^i x^i = 0$

then $\boxed{G = -u^A w_A}$

bosonic potential $\boxed{V_B^{\text{cl}} = \frac{1}{2}\delta^{ij} U_i U_j = 2(g^A w_{A,i})^2}$ with $\boxed{U = -2g^A w_A}$

then $\partial_i U_j - f_{ijk} U_k = 0$ flatness condition automatic!

integrability again $\boxed{u^{[A}_{\ i}\, u^{B]}_{\ ij} = 0 \quad \Leftrightarrow \quad u^A_{ij} + f_{ijk}\, u^A_k = 0}$ with $f = -u^{-1}\mathrm{d}u$

now $U\neq 0$: three- and four-particle solutions

construct permutation-invariant solutions for $n+1=3$ WDVV empty

homogeneous quadratic symmetric functions in (x,y,z):

$$u_1 = (x+y+z)^2\,, \qquad s=\frac{[(2x-y-z)(2y-z-x)(2z-x-y)]^2}{[(x-y)^2+(y-z)^2+(z-x)^2]^3}$$

$$u_2 = (x-y)^2+(y-z)^2+(z-x)^2\,,$$

$$u_3 = [(2x-y-z)(2y-z-x)(2z-x-y)]^{2/3}\,h(s)$$

functional freedom

compute $u_{A,i} \to w_{A,i} \to V_B^{\text{cl}},\ U,\ F$: integrability automatic

$$V_B^{\text{cl}} = \frac{g_1^2}{24u_1}+\frac{1}{324}\Big[(1-2s)g_2^2+2s\frac{(h\,g_2-g_3/\sqrt[3]{s})^2}{(h+3sh')^2}\Big]\Big(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\Big)$$

$$= \frac{g_1^2}{24u_1}+\frac{g_2^2-4s^{\frac{2}{3}-\delta}g_2g_3+2s^{\frac{1}{3}-2\delta}g_3^2}{324\,(1+3\delta)^2}\Big(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\Big)+\frac{\delta\,(2+3\delta)}{8\,(1+3\delta)^2}\frac{g_2^2}{u_2}$$

for the choice $\quad h(s)=s^{\delta} \quad\Leftrightarrow\quad u_3 = \dfrac{[(2x-y-z)(2y-z-x)(2z-x-y)]^{2/3+2\delta}}{[(x-y)^2+(y-z)^2+(z-x)^2]^{3\delta}}$

prepotentials:

$$U \;=\; -\tfrac{g_1}{6}\ln(x+y+z) - \frac{g_2}{18(1+3\delta)}\ln(x-y)(y-z)(z-x) - \frac{\delta\, g_2}{4(1+3\delta)}\ln u_2 + O(g_3)$$

$$\begin{aligned} F \;=\; & -\tfrac{1}{6}(x+y+z)^2\ln(x+y+z) \;+\; \tfrac{\delta}{4}\,u_2\ln u_2 \\ & -\tfrac{1}{4}\Big[(x-y)^2\ln(x-y) + (y-z)^2\ln(y-z) + (z-x)^2\ln(z-x)\Big] \\ & +\tfrac{1-6\delta}{36}\Big[(2x-y-z)^2\ln(2x-y-z)+(2y-z-x)^2\ln(2y-z-x)+(2z-x-y)^2\ln(2z-x-y)\Big] \end{aligned}$$

G_2 family plus ‘radial term’ $\sim$ $[(x-y)^2+(y-z)^2+(z-x)^2]\ln[(x-y)^2+(y-z)^2+(z-x)^2]$

special cases:

$\delta = 0 \quad \Leftrightarrow \quad h = 1 \quad : \quad V_B^{\text{cl}}(g_0{=}g_3{=}0)$ is pure Calogero

$\delta = \tfrac{1}{6} \quad \Leftrightarrow \quad h = s^{1/6} \quad : \quad V_B^{\text{cl}}(g_0{=}g_2{=}0)$ is pure Calogero

$V_B \;=\; V_B^{\text{cl}} + \frac{\hbar^2}{8}F'''F'''$ yields quantum corrections to the couplings g_i

$n{+}1=4$: WDVV active $\longrightarrow$ integrability nontrivial $\longrightarrow$ PDEs

take known WDVV solution F and solve $u^A_{ij} + f_{ijk}u^A_k = 0$ with $f = F'''$

hypergeometric ${}_2F_1$ appears in the few solutions u^A_i and $w_{A,i}$ we have found

example: based on A_3 solution with 'radial term'

$$F_{A_3\oplus A_1} = -\tfrac18 (\textstyle\sum_i x^i)^2 \ln|\textstyle\sum_i x^i| - \tfrac18 u_2 \ln u_2 + \tfrac18 \displaystyle\sum_{i<j} (x^i{-}x^j)^2 \ln|x^i{-}x^j|$$

we discovered

$$\begin{aligned} u_1 &= (x{+}y{+}z{+}w)^2 \\ u_2 &= (x{-}y)^2 + (x{-}z)^2 + (x{-}w)^2 + (y{-}z)^2 + (y{-}w)^2 + (z{-}w)^2 \\ u_3 &= u_2\, I(\tfrac{x+y-z-w}{p\,q}) \qquad \text{and} \qquad u_4 = u_2\, I(\tfrac{p}{q}) \end{aligned}$$

with
$$\begin{aligned} p^2 &= (x{-}y{+}z{-}w) + 2\sqrt{(w{-}x)(y{-}z)} \\ q^2 &= (x{-}y{-}z{+}w) + 2\sqrt{(w{-}y)(x{-}z)} \end{aligned} \qquad \text{and} \qquad I(x) = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^4}}$$

$\rightarrow$ $w_{A,i}$ and V_B^{cl} are algebraic but not of Calogero type also B_3 solution

Summary

- $\mathcal{N}{=}4$ superconformal n-particle mechanics in $d{=}1$ is governed by U and F
- U and F are subject to homogeneity, flatness and WDVV conditions
- covector ansatz for F leads to partial isometry conditions with multipliers λ_π
- finite Coxeter root systems and certain deformations thereof are solutions
- orthocentric polytope interpretation for certain solution families
- hypergraphs and matroids yield good candidates but don't guarantee WDVV
- geometric interpretation via flat superpotential $G \longrightarrow$ integrability conditions
- general 3-particle system constructed – 3 couplings and one free function
- higher-particle systems exist but tedious to construct $\longleftarrow$ hypergeometric