N=4 Supersymmetric Mechanics in Non-Abelian backgrounds

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Erevan, 2010

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- In the Hamiltonian approach we introduce isospin currents I^a which commute as

$$\left\{ I^a, I^b \right\}_{PB} = 2i\epsilon^{abc} I^c$$

- For the supersymmetric theories the natural framework is Lagrangian description in the superspace

Our goal is to construct the $N = 4$ supersymmetric mechanics describing the motion of isospin particles in the non-Abelian background.

The main questions are

- to which supermultiplet the isospin degrees of freedom belong to?
- how to write the corresponding action?

Clearly enough, in the Lagrangian approach one has to treat the isospin vector as a composite one. Straightforward realization:

- One has to add to the theory additional physical fermions and auxiliary bosons
- The Lagrangian for fermions has a standard form and coupling with physical bosons and fermions can be also done in a standard manner
- One may construct isospin currents I^a from the physical fermions presented in the theory as $I^a \sim (\psi\, \sigma^a \bar{\psi})$
- after quantization I^a will obey a proper algebra

In our approach we proposed to construct the isospin currents from bosonic variables as $I^a \sim (u\,\sigma^a \bar{u})$.

- These bosonic variables $u, \bar{u}$ have to enter the Lagrangian with first-order in time derivatives kinetic term as fermions ψ did

$$\sim \int dt \left(\dot{u}^i \bar{u}_i - u^i \dot{\bar{u}}_i \right), \qquad i = 1, 2.$$

  Thus, after quantization I^a will obey a proper algebra
- The fermions have to be auxiliary ones.

There are three different approaches:

- Put auxiliary bosons and fermions in gauge supermultiplet (S.Fedoruk, E.Ivanov, O.Lechtenfeld)
- Use reduction procedure (S.Bellucci, A.Sutulin, O.Lechtenfeld, S.K.)
  (in the bosonic case this procedure has been proposed by M. Gonzales, Z. Kuznetsova, A. Nersessian, F. Toppan, V. Yeghikyan)
- Use the specific coupling between auxiliary and "matter" supermultiplets
  (S.Bellucci, O.Lechtenfeld, A.Sutulin, S.K.)

We introduce the auxiliary fermionic $\Psi^{\hat\alpha}, \bar\Psi_{\hat\alpha}, \hat\alpha = 1,2$ superfields subjected to the following irreducible conditions

$$D^i\Psi^1 = 0, \quad D^i\Psi^2 + \overline{D}^i\Psi^1 = 0, \quad \overline{D}_i\Psi^2 = 0.$$

These constraints leave in the superfield Ψ four fermionic and four bosonic components

$$\psi^{\hat\alpha} = \Psi^{\hat\alpha}|, \qquad u^i = -D^i\bar\Psi^2|, \quad \bar u_i = \overline{D}_i\Psi^1|$$

Let us consider the following test action

$$S_t = -\frac{g}{32}\int dtd^4\theta\, \Psi^{\hat\alpha}\bar\Psi_{\hat\alpha}, \qquad g = const.$$

In terms of the components we will have

$$S_t = -g\int dt\left[\dot\psi^1\dot{\bar\psi}^2 - \dot\psi^2\dot{\bar\psi}^1 + \frac{i}{4}\left(\dot u^i\bar u_i - u^i\dot{\bar u}_i\right)\right]$$

The trick is to introduce new fermions as

$$\rho^{\hat\alpha} = \dot\psi^{\hat\alpha}, \quad \bar\rho_{\hat\alpha} = \dot\psi_{\hat\alpha}.$$

Now, the action has a proper form. More important thing is that despite the non-local definition of spinors $\rho^\alpha, \bar\rho_\alpha$ the action is invariant under following local N=4 supersymmetry transformations

$$\delta\rho^1 = -\bar\epsilon^i \dot{\bar u}_i,\ \delta\rho^2 = \epsilon_i \dot{\bar u}^i, \quad \delta u^i = -2i\epsilon^i \bar\rho^1 + 2i\bar\epsilon^i \bar\rho^2,\ \delta\bar u_i = -2i\epsilon_i\rho^1 + 2i\bar\epsilon_i\rho^2.$$

What about couplings with a matter supermultiplets? If we consider the following coupling of auxiliary Ψ superfields with some arbitrary, for time being, N=4 supermultiplet X

$$S_c = -\frac{1}{32}\int dt d^4\theta\ (X+g)\,\Psi^{\hat\alpha}\bar\Psi_{\hat\alpha},$$

then the replacement $\dot\psi \to \rho$ works if X obeys constraints

$$D^i D_i X = 0,\ \overline{D}_i \overline{D}^i X = 0, \quad \left[D^i, \overline{D}_i\right] X = 0$$

These constraints define $(1,4,3)$ supermultiplet with the components:

$$x = X|,\ A_{ij} = A_{(ij)} = \frac{1}{2}\left[D_i, \overline{D}_j\right] X|, \qquad \eta^i = -iD^i X|,\ \bar\eta_i = -i\overline{D}_i X|.$$

Thus, the first example of the N=4 mechanics with isospin degrees of freedom is given by the action

$$S = S_x + S_c = -\frac{1}{32}\int dtd^4\theta \mathcal{F}(X) - \frac{1}{32}\int dtd^4\theta\ (X+g)\,\Psi^{\hat\alpha}\bar\Psi_{\hat\alpha},$$

where $\mathcal{F}(X)$ is an arbitrary function of X.
After elimination of auxiliary fermions (in Ψ) and auxiliary bosons (in X) the bosonic part of the action acquires the form

$$S = \frac{1}{8}\int dt\left[F''\dot{x}^2 - 2i\left(\dot{w}^i\bar{w}_i - w^i\dot{\bar{w}}_i\right) - \frac{1}{F''x^2}\left(w^i\bar{w}_i\right)^2\right],\quad w^i = \sqrt{x}\,u^i$$

From equations of motion for w follows that

$$w^i\bar{w}_i = const$$

Therefore, the net effect from interaction with isospin degrees of freedom is appearance of the potential term.

It is important that with the choices

$$S^{Conf}_{\alpha} \sim \int dt d^4\theta \, (X)^{-\frac{1}{\alpha}} \,, \text{ and/or } \quad S^{Conf}_{-1} \sim \int dt d^4\theta X \log X,$$

our action is invariant under the $D(2,1;\alpha)$ group for arbitrary α.
It is interesting that four-fermions coupling in the case of $D(2,1;\alpha)$ invariant action reads

$$\frac{(1+\alpha)(1+2\alpha)}{64 y^2}\tilde{\eta}^2\bar{\tilde{\eta}}^2, \qquad \text{where } y = x^{-\frac{1}{2\alpha}},\ \tilde{\eta}^i = x^{-\frac{1}{2\alpha}-1}\frac{\eta^i}{\alpha},\ \alpha \neq -1, 0$$

Now it is clear that the simplest case of $N = 4$ superconformal invariant mechanics corresponds to the $\alpha = -1/2$ case, i.e. the $OSp(4|2)$ superconformal group, when the four-fermionic interaction disappears from the Lagrangian. This means that the corresponding supercharges contain the fermions only linearly, similarly to the $N = 2$ supersymmetric case.

It is clear that treating of the scalar bosonic superfield X as independent one is too restrictive, because the master constraints leave in this supermultiplet only one physical bosonic component x which is not enough to describe the isospin particle. The way to overcome this limitation is to treat the superfield X as a composite one, constructed from N=4 supermultiplets with bigger number of physical bosons. The two reasonable superfields from which it is possible to construct superfield X are N=4 tensor supermultiplet $\mathcal{V}^{ij}$ and a one-dimensional hypermultiplet $Q^{i\alpha}$.

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The N=4 tensor supermultiplet is described by the triplet of bosonic $N=4$ superfields $\mathcal{V}^{ij}=\mathcal{V}^{ji}$ subjected to the constraints

$$D^{(i}\mathcal{V}^{jk)}=\overline{D}^{(i}\mathcal{V}^{jk)}=0, \qquad \left(\mathcal{V}^{ij}\right)^{\dagger}=\mathcal{V}_{ij},$$

which leave in $\mathcal{V}^{ij}$ the following independent components:

$$v^a=-\frac{i}{2}\left(\sigma^a\right)_i{}^j\mathcal{V}^i_j|, \quad \lambda^i=\frac{1}{3}D^j\mathcal{V}^i_j|, \quad \bar{\lambda}_i=\frac{1}{3}\overline{D}_j\mathcal{V}^j_i|, \quad A=\frac{i}{6}D^i\overline{D}^j\mathcal{V}_{ij}|.$$

Thus its off-shell component field content is $(3,4,1)$, i.e. three physical v^a and one auxiliary A bosons and four fermions $\lambda^i,\bar{\lambda}_i$.
Now one may check that the composite superfield

$$X=\frac{1}{|\mathcal{V}|}\equiv\frac{1}{\sqrt{\mathcal{V}^a\mathcal{V}^a}},$$

where $\mathcal{V}^a=-\frac{i}{2}\left(\sigma^a\right)_i{}^j\mathcal{V}^i_j$, obeys needed constraints.

Due to composite structure, now all components of the X superfield, i.e. the physical boson x, fermions $\eta^i, \bar\eta_i$ and auxiliary fields A^{ij} are expressed through the components of $\mathcal{V}^{ij}$ supermultiplet as

$$x = \frac{1}{|v|}, \qquad \eta^i = \frac{v^a}{|v|^3}(\lambda\sigma^a)^i, \quad \bar\eta_i = \frac{v^a}{|v|^3}(\sigma^a\bar\lambda)_i,$$

$$A^i_j = -3\frac{v^a v^b}{|v|^5}(\lambda\sigma^a)^i(\sigma^b\bar\lambda)_j - \frac{v^a(\sigma^a)^i_j}{|v|^3}A + \frac{1}{|v|^3}\epsilon^{abc}v^a\dot v^b(\sigma^c)^i_j + \frac{1}{|v|^3}\left(\delta^i_j\lambda^k\bar\lambda_k - \lambda_j\bar\lambda^i\right).$$

The bosonic part of the action

$$S = S_v + S_c = -\frac{1}{32}\int dt d^4\theta \mathcal{F}(\mathcal{V}) + S_c,$$

where $\mathcal{F}(\mathcal{V})$ is now an arbitrary function of $\mathcal{V}^{ij}$, will contain the term

$$\sim \mathcal{A}_a\dot v_a, \quad \text{where }, \quad \mathcal{A}_a = -\frac{i}{2v^2}\epsilon_{abc}v_b I_c, \;\; I_a = \frac{i}{2}(w\sigma_a\bar w).$$

This is just the potential of the Wu-Yang monopole.

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Another option is to choose as the basic superfields a quartet of real $N=4$ superfields Q^{ia}, $(i,a=1,2)$ defined in $N=4$ superspace and obeying to the constraints

$$D^{(i}Q^{j)\alpha}=0,\quad \overline{D}^{(i}Q^{j)\alpha}=0,\qquad \left(Q^{i\alpha}\right)^{\dagger}=Q_{i\alpha}$$

This $N=4$ supermultiplet describes four bosonic and four fermionic fields and no auxiliary fields off-shell:

$$q^{i\alpha}=\mathcal{Q}^{i\alpha}|,\qquad \eta^{i}=-iD^{i}\left(\frac{2}{\mathcal{Q}^{j\alpha}\mathcal{Q}_{j\alpha}}\right)|,\ \bar{\eta}_i=-i\overline{D}_i\left(\frac{2}{\mathcal{Q}^{j\alpha}\mathcal{Q}_{j\alpha}}\right)|$$

Now one may define the composite superfield X which obey to the needed constraints as

$$X=\frac{2}{Q^{i\alpha}Q_{i\alpha}}.$$

The former auxiliary components A_{ij} expressed via the components of $\mathcal{Q}^{i\alpha}$ as

$$A_{ij}=-\frac{4i}{(q^{k\beta}q_{k\beta})^2}\left(\dot{q}_i^{\alpha}q_{j\alpha}+\dot{q}_j^{\alpha}q_{i\alpha}\right)-\frac{(q^{k\beta}q_{k\beta})}{2}\left(\eta_i\bar{\eta}_j+\eta_j\bar{\eta}_i\right).$$

Like to the case of tensor supermultiplet one may write the full action with hypermultiplet self-interacting part S_q added as

$$S = S_q + S_c = -\frac{1}{32}\int dtd^4\theta\mathcal{F}(\mathcal{Q}) + S_c,$$

where now $\mathcal{F}(\mathcal{Q})$ is an arbitrary function of $\mathcal{Q}^{i\alpha}$. This action describes the motion of an isospin particle on a conformally flat four-manifold carrying the non-Abelian field of a BPST instanton.

While dealing with tensor supermultiplet $\mathcal{V}^{ij}$ and hypermultiplet $\mathcal{Q}^{i\alpha}$ the structure of the action S_c can be generalized to be

$$S_c = -\frac{1}{32}\int dtd^4\theta\; Y\Psi^{\hat\alpha}\bar\Psi_{\hat\alpha},$$

with Y obeying

$$\triangle_3 Y = 0 \text{ (tensor supermultiplet)},\; \triangle_4 Y = 0 \text{ (hypermultiplet)}.$$

Clearly, our choices $Y = X + g$ with X defined as $\frac{1}{|v|}$ and $\frac{1}{q^2}$, respectively, correspond to spherically-symmetric solutions of 3D and 4D Laplace equations.

One of the most attractive features of our approach is the unified structure of the action S_c, which has the same form for any type of supermultiplets which we are using to construct a composite superfield X. Just this opens the way to relate the different systems via duality transformations.
Indeed, it is known for a long time that in one dimension one may switch between supermultiplets with a different number of physical bosons by expressing the auxiliary components through time derivative of physical bosons, and vice versa:

$$A \to \dot{\phi}, \qquad \dot{\phi} \to A.$$

Here we will use this mechanism to obtain the action of tensor multiplet from the hypermultiplet action and then, alternatively, we will reobtain the action for hypermultiplet (with some restrictions) starting from action for tensor one.
Useful parametrization:

$$q^{11} = \frac{e^{\frac{1}{2}(u-i\phi)}}{\sqrt{1+\Lambda\bar{\Lambda}}}\Lambda, \quad q^{21} = -\frac{e^{\frac{1}{2}(u-i\phi)}}{\sqrt{1+\Lambda\bar{\Lambda}}}, \quad q^{22} = \left(q^{11}\right)^{\dagger}, \quad q^{21} = -\left(q^{12}\right)^{\dagger},$$

$$V^{11} = 2i\frac{e^{u}}{1+\Lambda\bar{\Lambda}}\Lambda, \qquad V^{22} = -2i\frac{e^{u}}{1+\Lambda\bar{\Lambda}}\bar{\Lambda}, \qquad V^{12} = -ie^{u}\left(\frac{1-\Lambda\bar{\Lambda}}{1+\Lambda\bar{\Lambda}}\right).$$

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The "auxiliary" components A^{ij} expressed in terms of V^{ij} and, alternatively, in terms of $Q^{i\alpha}$. Identifying both expressions one may find that

$$A = i\left(\dot{q}^{i1} q_i^2 + \dot{q}^{i2} q_i^1\right) + \frac{1}{4}(q^{k\alpha} q_{k\alpha})^2\, q^{i1} q^{j2}\, (\eta_i \bar{\eta}_j + \eta_j \bar{\eta}_i)\,,$$

or, in another components

$$\dot{\phi} = e^{-u} A - i\frac{\dot{\Lambda}\bar{\Lambda} - \Lambda\dot{\bar{\Lambda}}}{1 + \Lambda\bar{\Lambda}} - \frac{1}{4} e^{-u} (q^{k\alpha} q_{k\alpha})^2\, q^{i1} q^{j2}\, (\eta_i \bar{\eta}_j + \eta_j \bar{\eta}_i)\,.$$

Thus, we see that to get the action for the tensor supermultiplet one has to replace in the component action for the hypermultiplet the time derivative of the field ϕ by the combination which includes the new auxiliary field A. The additional restriction comes from S_q part of the action which has depend now only on tensor supermultiplet. If it is so, then in the full action the field ϕ will enter only through $\dot{\phi}$ and the discussed replacement will be valid.

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It is clear that the backward procedure also exists. Indeed, from expressions for auxiliary components A^{ij} one may get :

$$A = \frac{1}{f}\left[\dot{\phi} + \frac{\partial}{\partial v_a} f(\lambda\sigma^a\bar{\lambda}) - a_a \dot{v}_a\right],$$

where

$$f = \frac{1}{|v|}, \quad \text{and}\,, \quad a_1 = -\frac{v_2(v_3+|v|)}{(v_1^2+v_2^2)|v|},\ a_2 = \frac{v_1(v_3+|v|)}{(v_1^2+v_2^2)|v|},\ a_3 = 0.$$

Thus, to get the hypermultiplet action from those one for tensor supermultiplet one has to dualize the auxiliary component A into new physical boson ϕ using above formula. Of course, we do not expect to get the most general action for the hypermultiplet interacting with isospin-containing supermultiplet Ψ because S_v part depends only on $\mathcal{V}$ supermultiplet. But we, for sure, will get a particular class of hypermultiplet actions with one isometry with the Killing vector ∂_ϕ.

Our consideration has one subtle point. Indeed, if we rewrite the relation between auxiliary component A (in tensor supermultiplet) and physical boson ϕ (in hypermultiplet) as

$$\dot{\phi} = a_a \dot{v}_a - f_{,a}(\lambda\sigma^a\bar{\lambda}) + fA, \qquad f_{,a} \equiv \frac{\partial}{\partial v_a} f,$$

then the r.h.s. of this equation has to transform as the full time derivative under supersymmetry transformations. One may check that it is so if f and a_a are chosen as above. But this choice in not unique. It has been proved in A. Shcherbakov & S.K., Phys.Lett. B637(2006)119, that the r.h.s. of this equation transforms as the full time derivative if the function f and a_a satisfy the equations

$$\triangle_3 f = 0, \qquad rot\ \vec{a} = \vec{\nabla} f.$$

Only two solutions $f_1 = \frac{1}{|v|}$ and $f_2 = const$ are related with hypermultiplet!

Thus, the general scheme includes the following steps

- We start from the action for vector supermultiplet interacting with Ψ

$$S = -\frac{1}{32}\int dt d^4\theta \left(\mathcal{F}(\mathcal{V}) + (g + \frac{1}{|\mathcal{V}|})\Psi^{\hat{\alpha}}\bar{\Psi}_{\hat{\alpha}} \right)$$

- In the component action we replace the auxiliary component A as

$$A = \frac{1}{f}\left[\dot{\phi} + \frac{\partial}{\partial v_a} f(\lambda\sigma^a\bar{\lambda}) - a_a\dot{v}_a\right],$$

- (Optional step) We will choose prepotential $\mathcal{F}$ as

$$\triangle_3 \mathcal{F}| = f$$

The resulting action will contain

- The bosonic kinetic term

$$S_k = \int dt \left[f \dot{v}_a \dot{v}_a + \frac{1}{f} (\dot{\phi} + a_b \dot{v}_b)^2 \right]$$

- Interaction with monopole field

$$S_i = \int dt \left[\frac{4i}{(1+g|v|)|v|^2} v^a I^c \left(\frac{1}{f} \delta^{ac} (-\dot{\phi} + a_b \dot{v}_b) + \epsilon^{abc} \dot{v}^c \right) \right]$$

With the conditions

$$\triangle_3 f = 0, \qquad rot\, \vec{a} = \vec{\nabla} f.$$

this action describes Hyper-Kahler sigma model. Three simplest solutions include

- Flat space: $f = const$ and $f = \frac{1}{|v|}$.
- Taub-NUT model: $f = 1 + \frac{\kappa}{|v|}$

Taub-NUT

One center Taub-NUT metrics is defined by

$$f = p_1 + \frac{p_2}{|v|}, \qquad p_1, p_2 = \text{const, we will choose } f = g + \frac{1}{|v|}.$$

With such a definition f coincides with the function $Y = g + \frac{1}{|v|}$. To get the Taub-NUT metrics, one has also to fix the pre-potential F to be equal to f. Thus, we have

$$S = \frac{1}{8}\int dt \left[\left(g + \frac{1}{|v|}\right)\dot{v}_a\dot{v}_a + \frac{1}{\left(g+\frac{1}{|v|}\right)}\left(\dot{\phi} - a_a\dot{v}_a\right)^2 + i\left(\dot{\xi}^i\bar{\xi}_i - \xi^i\dot{\bar{\xi}}_i\right) - 2i\left(\dot{w}^i\bar{w}_i - w^i\bar{w}_i\right)\right.$$

$$+\frac{i}{(1+g|v|)|v|^2}\left[\frac{v_a}{\left(g+\frac{1}{|v|}\right)}\left(\dot{\phi} - a_c\dot{v}_c\right) - \epsilon_{abc}v_b\dot{v}_c\right](\Sigma_a - 4I_a)$$

$$\left. + \frac{4(1+3g|v|)}{(1+g|v|)^3|v|^3}\left(v_aI_a\right)\left(v_b\Sigma_b\right) - \frac{4g}{(1+g|v|)^3}\left(I_a\Sigma_a\right)\right].$$

The bosonic term in the second line of this action can be rewritten as

$$\mathcal{A}_a I_a = \frac{i}{2}\left[\frac{1}{f}\,\frac{\partial \log f}{\partial v_a}\left(\dot{\phi} - a_c \dot{v}_c\right) - \epsilon_{abc}\frac{\partial \log f}{\partial v_b}\dot{v}_c\right] I_a,$$

where f is defined above as

$$f = g + \frac{1}{|v|}.$$

In this form the vector potential $\mathcal{A}_a$ coincides with the potential of a Yang-Mills $SU(2)$ instanton in the Taub-NUT space, if we may view I_a, as defined

$$I^a = \frac{i}{2}\left(w\sigma^a\bar{w}\right), \qquad \Sigma^a = -i\left(\xi\sigma^a\bar{\xi}\right),$$

as proper isospin matrices. The remaining terms in the second and third lines of the action provide a N=4 supersymmetric extension of the instanton.

We have proposed the Lagrangian formulations of $N = 4$ supersymmetric four-dimensional isospin-carrying particle moving in Hyper-Kahler space in the presence of non-Abelian background gauge field. The isospin degrees of freedom are described on the Lagrangian level by bosonic auxiliary variables forming $N = 4$ supermultiplet with additional, also auxiliary fermions.

Interesting questions

- The structure of background gauge field: is this really field of some monopole?
- Hamiltonian and Supercharges: four-fermions coupling is absent in the case of HK metrics!
- Existence of the conserved Runge-Lenz vector
- More examples of HK metrics (Eguchi-Hanson?)
- Quantization