Harmonic Superspace in One Dimension: Applications in $\mathcal{N} = 4$ SQM

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Basic $\mathcal{N}=4, d=1$ multiplets

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$\mathcal{N}=4, 4D$ SQM with non-abelian gauge fields

$\mathcal{N}=4, 3D$ SQM in non-abelian monopole background

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SQM is the simplest ($d = 1$) supersymmetric theory:

- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Extended SUSY in $d = 1$ is specific: dualities between various supermultiplets (Gates Jr. & Rana, 1995, Pashnev & Toppan, 2001) nonlinear “cousins” of off-shell linear multiplets (E.I., S.Krivonos, O.Lechtenfeld, 2003, 2004), etc.

The efficient tool to deal with extended supersymmetries in $d > 1$ is Harmonic Superspace (A. Galperin, E.I., V. Ogievetsky, E. Sokatchev, 1984):

- Off-shell formulation of hypermultiplets in $\mathcal{N} = 2, d = 4$ and $\mathcal{N} = 1, d = 6$ SUSY;
- Formulation of $\mathcal{N} = 4, d = 4$ SYM with the maximal number $\mathcal{N} = 3$ of off-shell supersymmetries;
- ETC...

Motivations

What about applications in $d = 1$ theories? The $\mathcal{N} = 4, d = 1$ version of HSS was elaborated in E.I., O. Lechtenfeld, 2003:

- The powerful device of $\mathcal{N} \geq 4$ SQM model-building;
- Allows to understand interrelations between various $\mathcal{N} = 4$ *SQM* models via the manifestly $\mathcal{N} = 4$ covariant gauging procedure (F. Delduc, E.I., 2006, 2007);
- Provides new $\mathcal{N} = 4$ superextensions of Calogero-type models (S. Fedoruk, E.I., O. Lechtenfeld, 2008 - 2010);
- Latest development - applications to SQM models with the Lorentz-force type couplings to the external gauge field $\sim A_m(x(t))\dot{x}^m(t)$.

Motivations

Why SQM models with external gauge fields are of interest:

- $d = 1$ prototype of the p-branes world-volume couplings;
- Supersymmetric versions of Wilson loops, Berry phase;
- Superextensions of the Landau problem and of the quantum Hall effect;
- Quantum-mechanical realizations of Hopf maps (e.g., Gonzales, Kuznetsova, *et al*, 2009);
- Superextensions of Chern-Simons mechanics (Howe & Townsend, 1990).

- Until recently - $\mathcal{N}=4$ superextensions only for the abelian background gauge fields. $\mathcal{N}=4$ extension - in $\mathcal{N}=4, d=1$ HSS (Ivanov, Lechtenfeld, hep-th/0307111).
- $\mathcal{N}=2, 3D$ SQM models in a background of non-abelian monopoles were considered e.g. in Feher, Horvathy, O'Raifeartaigh, 1989.
- The coupling to non-abelian gauge background in $\mathcal{N}=4$ case: Konyushikhin, Smilga, 0910.5162 [hep-th], Ivanov, Konyushikhin, Smilga, 0912.3289 [hep-th], Ivanov, Konyushikhin, 1004.4597 [hep-th].
- Exploits the semi-dynamical (or spin) supermultiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ (Fedoruk, Ivanov, Lechtenfeld, 0812.4276 [hep-th]).
- Bosonic fields of the spin multiplet are described by the U(1) gauged Wess-Zumino $d=1$ action $\Rightarrow$ generators of the gauge group SU(2) upon quantization. Manifestly $\mathcal{N}=4$ supersymmetric $d=1$ gauging of isometries: Delduc, Ivanov, hep-th/0605211.
- Off-shell $\mathcal{N}=4$ SUSY requires the external gauge potential to be (a) self-dual and (b) satisfying the $4D$ 't Hooft ansatz or its $3D$ reduction.
- On-shell $\mathcal{N}=4$ SUSY: compatible with the general self-dual background (Konyushikhin, Smilga, 0910.5162 [hep-th]).

Harmonic $\mathcal{N}=4, d=1$ superspace

- Ordinary $\mathcal{N}=4, d=1$ superspace:

$$(t, \theta^\alpha, \bar\theta_\alpha), \quad \alpha = 1,2;$$

- Harmonic extension:

$$(t, \theta^\alpha, \bar\theta_\alpha) \quad \Rightarrow \quad (t, \theta^\alpha, \bar\theta_\alpha, u^\pm_\alpha),\; u^{+\alpha}u^-_\alpha = 1,\; u^\pm_\alpha \in SU(2)_{Aut}.$$

- Analytic basis:

$$(t_A, \theta^+, \bar\theta^+, u^\pm_\alpha, \theta^-, \bar\theta^-) \equiv (\zeta, u^\pm, \theta^-, \bar\theta^-)$$

$$\theta^\pm = \theta^\alpha u^\pm_\alpha,\; \bar\theta^\pm = \bar\theta^\alpha u^\pm_\alpha,\;\; t_A = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)$$

- Analytic superspace and superfields:

$$D^+ = \frac{\partial}{\partial\theta^-},\; \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}, \quad D^+\Phi = \bar D^+\Phi = 0 \;\Rightarrow\; \Phi = \Phi(\zeta, u^\pm)$$

- Harmonic derivatives:

$$D^{\pm\pm} = u^\pm_\alpha \frac{\partial}{\partial u^\mp_\alpha} + \theta^\pm \frac{\partial}{\partial\theta^\mp} + \bar\theta^\pm \frac{\partial}{\partial\bar\theta^\mp} + 2i\theta^\pm\bar\theta^\pm \frac{\partial}{\partial t_{\rm A}},$$

$$[D^+, D^{++}] = [\bar D^+, D^{++}] = 0 \quad \Rightarrow \quad D^{++}\Phi(\zeta, u^\pm) \text{ is analytic}$$

Basic $\mathcal{N}=4, d=1$ multiplets

1. The multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ - $q^{+a}(\zeta,u) \propto (x^{\alpha a}, \chi^a, \bar\chi^a),\ a=1,2$:

$$D^{++}q^{+a}=0, \quad q^{+a}=x^{\alpha a}u^+_\alpha - 2\theta^+\chi^a - 2\bar\theta^+\bar\chi^a - 2i\theta^+\bar\theta^+\dot x^{\alpha a}u^-_\alpha\,,$$

$$S_{free} \sim \int dtd^4\theta du\, q^{+a}D^{--}q^+_a \sim \int dt\left(\dot x^{\alpha a}\dot x_{\alpha a} + i\bar\chi^a\dot\chi_a\right).$$

2. The multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ - $L^{++}(\zeta,u) \propto (\ell^{(\alpha\beta)}, \psi^\alpha, \bar\psi^\alpha, F)$:

$$D^{++}L^{++}=0,\ L^{++}=\ell^{\alpha\beta}u^+_\alpha u^+_\beta + i(\theta^+\chi^\alpha+\bar\theta^+\bar\chi^\alpha)u^+_\alpha + \theta^+\bar\theta^+(F-2i\dot\ell^{\alpha\beta}u^+_\alpha u^-_\beta),$$

$$S_{free} \sim \int dtd^4\theta du\, L^{++}(D^{--})^2L^{++} \sim \int dt\left[\left(\dot\ell^{\alpha\beta}\dot\ell_{\alpha\beta} - \frac{1}{2}F^2\right) + i\bar\psi^\alpha\dot\psi_\alpha\right].$$

3. Gauge multiplet - $V^{++}(\zeta,u)$:

$$V^{++\prime}=V^{++}+D^{++}\Lambda,\ \Lambda=\Lambda(\zeta,u)\ \Rightarrow\ V^{++}_{WZ}=2i\theta^+\bar\theta^+B(t)\,,\ \delta B=\dot\lambda(t).$$

4. Gauged $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet - $(v^+,\bar v^+)$, $\quad v^{+\prime}=e^{i\Lambda}v^+,\ \bar v^{+\prime}=e^{-i\Lambda}\bar v^+$:

$$(D^{++}+iV^{++})v^+=0,\ v^+=\phi^\alpha u^+_\alpha+\theta^+\omega_1+\bar\theta^+\bar\omega_2-2i\theta^+\bar\theta^+(\dot\phi^\alpha+iB\phi^\alpha)u^-_\alpha\,.$$

A simple example of $d = 1$ gauging in bosonic system

- Consider a complex $d = 1$ field $z(t), \bar{z}(t)$ with the following Lagrangian:

$$L_0 = \dot{z}\,\dot{\bar{z}} + i\kappa\left(\dot{z}\bar{z} - z\dot{\bar{z}}\right)$$

 The first term is the kinetic energy, the second one is the simplest example of Wess-Zumino term. One of the symmetries of this system is the $U(1)$ invariance under

$$z' = e^{-i\lambda} z\,, \quad \bar{z}' = e^{i\lambda}\bar{z}\,.$$

- Gauge this symmetry by promoting $\lambda \to \lambda(t)$. The gauge invariant action involves the $d = 1$ gauge field $A(t)$

$$L_{gauge} = (\dot{z} + iAz)\,(\dot{\bar{z}} - iA\bar{z}) + i\kappa\left(\dot{z}\bar{z} - z\dot{\bar{z}} + 2iAz\bar{z}\right) + 2c\,A\,,\ A' = A + \dot{\lambda}\,,$$

 where a “Fayet-Iliopoulos term” $\sim c$ has been added. It is gauge invariant (up to a total derivative) by itself.

- The next step is to choose the appropriate gauge in L_{gauge}:

$$z = \bar{z} \equiv \rho(t)$$

- Substitute it into L_{gauge} and obtain:

$$L_{gauge} = (\dot{\rho} + iA\rho)(\dot{\rho} - iA\rho) + 2i\kappa A\rho^2 + 2c\,A = (\dot{\rho})^2 + A^2\rho^2 - 2\kappa A\rho^2 + 2cA\,.$$

- The field $A(t)$ is the typical example of auxiliary field: it can be eliminated by its algebraic equation of motion:

$$\delta A: \quad A = \kappa - \frac{c}{\rho^2}\,.$$

- Finally obtain

$$L_{gauge} \Rightarrow (\dot{\rho})^2 - \left(\kappa\rho - \frac{c}{\rho}\right)^2.$$

This is a one-particle prototype of Calogero-Moser system. At $\kappa = 0$, recover the standard conformal mechanics:

$$L^{(c=0)}_{gauge} = (\dot{\rho})^2 - \frac{c^2}{\rho^2}\,.$$

- This gauging procedure can be interpreted as an off-shell Lagrangian analog of the well known Hamiltonian reduction.

Gauging in $\mathcal{N} = 4, d = 1$ HSS

- Example: start from the free action of the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$,

$$S = \int dt d^4\theta q^{+a} D^{--} q_a^+ \,, \quad \text{invariant under } \; q^{+a} \to q^{+a} + \lambda u^{+a} \,, \; a = 1, 2.$$

- Gauging: $\lambda \to \Lambda(\zeta, u)$

$$D^{++} q^{+a} = 0 \;\to\; \nabla^{++} q^{+a} = D^{++} q^{+a} - V^{++} u^{+a} = 0 \,,$$

$$S \to S_g = \int dt d^4\theta \, q^{+a} \nabla^{--} q_a^+ \,, \quad \nabla^{--} q_a^+ = D^{--} q^{+a} - V^{--} u^{+a} \,,$$

$$[\nabla^{++}, \nabla^{--}] = D^0 \;\Rightarrow\; D^{++} V^{--} - D^{--} V^{++} = 0 \,, \; V^{--} = V^{--}(V^{++}, u) \,.$$

- Gauge choice: $u^{-a} q_a^+ = 0 \;\Rightarrow\; q^{+a} = u^{-a} L^{++}$

$$D^{++} q^{+a} - V^{++} u^{+a} = 0 \;\Rightarrow\; V^{++} = L^{++} \,, \; D^{++} L^{++} = 0 \,,$$

$$D^{++} V^{--} - D^{--} L^{++} = 0 \;\Rightarrow\; V^{--} = \frac{1}{2} (D^{--})^2 L^{++} \,,$$

$$S_g = \int dt d^4\theta \, V^{--} L^{++} = \frac{1}{2} \int dt d^4\theta \, L^{++} (D^{--})^2 L^{++} \,.$$

- Thus arrive at the free action of the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$!

Gauging in $\mathcal{N} = 4, d = 1$ HSS

Everything works perfectly also for the interaction case and for other $\mathcal{N} = 4, d = 1$ multiplets. These multiplets and their superfield actions can be reproduced as the appropriate gaugings of the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and of some nonlinear generalizations of the latter.

- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$ linear $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ - via gauging shifting or rotational U(1) symmetry of q^{+a};
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$ non-linear $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ - via gauging scale symmetry, $q^{+a\prime} = \lambda\, q^{+a}$;
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow (\mathbf{2}, \mathbf{4}, \mathbf{2})$ - via gauging some two-generator solvable symmetry of q^{+a};
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow (\mathbf{1}, \mathbf{4}, \mathbf{3})$ - via gauging SU(2), $q^{+a\prime} = \lambda^a_b\, q^{+b}$;
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow (\mathbf{0}, \mathbf{4}, \mathbf{4})$ - via gauging the semi-direct product of SU(2) and shift $\delta q^{+a} = \lambda u^{+a}$.

$\mathcal{N}=4, 4D$ SQM with abelian external gauge field

The $\mathcal{N}=4$ SQM with coupling of $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet to the background gauge field is described by the action (E.I., O.Lechtenfeld, 2003):

$$S = \int dt d^4\theta du\, R_{kin}(q^{+a}, D^{--}q^{+b}, u) + \int du d\zeta^{(-2)}\, \mathcal{L}^{+2}(q^{+a}, u) \equiv S_1 + S_2$$

The second term involves only one time derivative on the bosonic field $x^{\alpha a}$ and so is an example of $d=1$ Wess-Zumino term

$$S_2 \sim \int dt \left(\mathcal{A}_{\alpha b}(x)\dot{x}^{\alpha b} + fermions\right), \; \mathcal{A}_{\alpha b}(x) = \int du\, u^-_\alpha \frac{\partial \mathcal{L}^{+2}}{\partial q^{+b}}|_{\theta=0}\,,$$

$$\mathcal{F}_{\alpha b\,\beta d} = \partial_{\alpha b}\mathcal{A}_{\beta d} - \partial_{\beta d}\mathcal{A}_{\alpha b} = \epsilon_{\alpha\beta}\mathcal{F}_{(bd)}\,, \quad \mathcal{F}_{(\alpha\beta)} = 0 - \text{Self-duality condition}$$

Thus $\mathcal{N}=4$ SUSY requires the external gauge field to be self-dual. No such a requirement is implied by $\mathcal{N}=2, d=1$ SUSY, e.g.

How to extend this to the most interesting non-abelian case?

Non-abelian self-dual background

- Adding the "spin" multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ (E.I., M.Konyushikhin, A.Smilga, 2009):

$$S = \int dtd^4\theta du\, R_{kin}(q^{+a}, D^{--}q^{+b}, u) - i\frac{k}{2}\int du d\zeta^{(-2)}\, V^{++}$$
$$-\frac{1}{2}\int du d\zeta^{(-2)}\, K(q^{+a}, u) v^{+}\bar{v}^{+} \equiv S_1 + S_2 + S_3\,,$$
$$(D^{++} + iV^{++})v^{+} = (D^{++} - iV^{++})\bar{v}^{+} = 0\,.$$

- S_1 describes a sigma-model type interaction of $x^{\alpha a}$:

$$S_1 \sim \int dt\left(f^{-2}(x)\dot{x}^{\alpha a}\dot{x}_{\alpha a} + fermions\right), \; f^{-2}(x) - \text{conformal factor.}$$

- S_2 - one-dimensional "Fayet-Iliopoulos" term:

$$S_2 = k\int dt\, B\,.$$

- S_3 - generalized Wess-Zumino term:

$$S_3 \sim \int dt\left[i\bar{\varphi}^{\alpha}(\dot{\varphi}_{\alpha} + iB\varphi_{\alpha}) - \frac{1}{2}\bar{\varphi}^{\beta}\varphi_{\gamma}\,(\mathcal{A}_{\alpha a})_{\beta}{}^{\gamma}\,\dot{x}^{\alpha a} + \text{fermions without } \partial_t\right],$$

$$(\mathcal{A}_{\alpha b})_{\beta}{}^{\gamma} = \frac{i}{h}\left(\varepsilon_{\alpha\beta}\,\partial^{\gamma}{}_{b}h - \frac{1}{2}\delta^{\gamma}_{\beta}\,\partial_{\alpha b}h\right), \quad h(x) = \int du\, K(x^{+a}, u^{\pm}_{\beta}),\; \Box h(x) = 0\,.$$

Non-abelian self-dual background (cont.)

- The above $\mathcal{A}_{\alpha b}$ is self-dual, $\mathcal{F}_{\alpha\beta} = 0$. It precisely matches with the general 't Hooft ansatz for $4D$ self-dual SU(2) gauge fields:

$$(\mathcal{A}_{\alpha b})^{\beta}_{\gamma} \Rightarrow (\mathcal{A}_{\mu})^{\beta}_{\gamma} = \frac{1}{2}\mathcal{A}^{i}_{\mu}(\sigma_i)^{\beta}_{\gamma},$$

$$\mathcal{A}^{i}_{\mu} = -\bar{\eta}^{i}_{\mu\nu}\partial_{\nu}\ln h(x)\,, \quad \bar{\eta}^{k}_{ij} = \varepsilon_{kij},\, \bar{\eta}^{k}_{0i} = -\bar{\eta}^{k}_{i0} = \delta_{ki} \quad (i,j,k = 1,2,3)\,.$$

- Example: one-instanton configuration on S^4:

$$ds^2 = \frac{4R^4 dx^2_{\mu}}{(x^2+R^2)^2}\,, \quad \mathcal{A}^{i}_{\mu} = \frac{2R^2\bar{\eta}^{i}_{\mu\nu}x_{\nu}}{x^2(x^2+R^2)}$$

- Can be brought to the BPST form, $\hat{\mathcal{A}}^{i}_{\mu} = \frac{2\eta^{i}_{\mu\nu}x_{\nu}}{x^2+R^2}\,,\ \hat{\mathcal{F}}^{a}_{\mu\nu} = -\frac{4R^2\eta^{a}_{\mu\nu}}{(x^2+R^2)^2}$, by the gauge transformation

$$\mathcal{A}_{\mu} \rightarrow \hat{\mathcal{A}}_{\mu} = U^{\dagger}\mathcal{A}_{\mu}U + iU^{\dagger}\partial_{\mu}U\,, \quad U(x) = -i\sigma_{\mu}x_{\mu}/\sqrt{x^2}.$$

  Corresponds to the following choice of the functions $K(x^{+a}, u)$ and $h(x)$:

$$K(x^{+a}, u^{\pm}_{\beta}) = 1 + \frac{1}{\left(c^{-}_{A}x^{+a}\right)^2}\,, \quad h(x) = 1 + \frac{R^2}{x^2_{\mu}},\ c^{-a} = c^{\alpha a}u^{-}_{\alpha}\,,\ R^2 = |c|^{-2}.$$

$\mathcal{N}=4$ SQM with Yang monopole

- As by-product - $\mathcal{N}=4$ SQM with Yang monopole:

$$L_{\mathbb{R}^5} = \frac{1}{2}\left(\dot{y}_5\dot{y}_5 + \dot{y}_\mu\dot{y}_\mu\right) + \mathcal{B}^i_\mu(y)\frac{1}{2}(\bar{\varphi}\sigma^i\varphi)\,\dot{y}_\mu, \quad \mu = 1,2,3,4\,,$$

$$\mathcal{B}^i_\mu = \frac{\eta^i_{\mu\nu}y_\nu}{r(r+y_5)}\,, \qquad r = \sqrt{y_5^2 + y_\mu^2}\,,$$

- After the polar decomposition of $\mathbb{R}^5$ into $\mathbb{S}^4 \sim \{\tilde{y}_\mu\}$ and radius r as

$$(y_5, y_\mu) \;\rightarrow\; (r, \sqrt{1-\tilde{y}_\mu^2}, \tilde{y}_\mu)\,,$$

and passing to the stereographic-projection coordinates as

$$\tilde{y}_\mu = 2\,\frac{x_\mu}{1+x^2}\,,$$

we get

$$L_{\mathbb{R}^5} = \frac{1}{2}\left\{\dot{r}^2 + 4r^2\frac{\dot{x}_\mu\dot{x}_\mu}{(1+x^2)^2}\right\} + \frac{2\eta^a_{\mu\nu}x_\nu\dot{x}_\mu\,\frac{1}{2}(\bar{\varphi}\sigma^a\varphi)}{1+x^2}\,.$$

- Thus the $5D$ mechanics with the gauge coupling to Yang monopole and "frozen" radial coordinate r admits extension to $\mathcal{N}=4$ SQM model.

Quantization of spin variables

- The relevant part of the total action:

$$S = \int dt \left[i\bar{\varphi}^{\alpha}(\dot{\varphi}_{\alpha} + iB\varphi_{\alpha}) \; + \; kB + \mathcal{A}^{i}_{\mu} T^{i} \dot{x}_{\mu} \right], \quad T^{i} = \frac{1}{2}\bar{\varphi}^{\alpha} \left(\sigma^{i}\right)_{\alpha}^{\;\beta} \varphi_{\beta},,$$

  $k =$ integer - from the requirement of invariance of the Euclidean path integral under topologically non-trivial gauge transformations (Polychronakos, 1991):

$$B(t) \to \; B(t) + \dot{\alpha}(t), \qquad \varphi(t) \to e^{-i\alpha(t)}\varphi(t)$$

- Constraint - by varying w.r.t. $B(t)$: $\quad \bar{\varphi}^{\alpha}\varphi_{\alpha} = k \, .$
- Quantization by Dirac:

$$[\varphi_{\alpha}, \bar{\varphi}^{\beta}] = \delta^{\beta}_{\alpha} \, , \quad [\varphi_{\alpha}, \varphi_{\beta}] = [\bar{\varphi}^{\alpha}, \bar{\varphi}^{\beta}] = 0, \; \varphi_{\alpha} \to \partial/\partial\bar{\varphi}^{\alpha} \, ,$$

$$\bar{\varphi}^{\alpha}\varphi_{\alpha}\Psi = \bar{\varphi}^{\alpha}\frac{\partial}{\partial\bar{\varphi}^{\alpha}}\Psi = k\Psi \, .$$

- Wave functions - homogeneous polynomials of $\bar{\varphi}^{\alpha}$ of degree k .

SU(2) multiplet structure of wave function

$$T^i \to T^i = \frac{1}{2}\bar{\varphi}^\alpha \left(\sigma^i\right)_\alpha^{\ \beta} \frac{\partial}{\partial\varphi^\beta}, \quad [T^i, T^k] \ = \ i\varepsilon^{ikl} T^l$$

$$T^i T^i \ = \ \frac{1}{4}\left[(\bar{\varphi}^\alpha \varphi_\alpha)^2 + 2(\bar{\varphi}^\alpha \varphi_\alpha)\right] \ = \ \frac{k}{2}\left(\frac{k}{2}+1\right).$$

Thus T^i are generators of SU(2) in the irrep of spin $k/2$. An interesting feature is that this gauge SU(2) is R-symmetry group of $\mathcal{N} = 4$ SUSY.

$\mathcal{N} = 4$ SQM in non-abelian monopole background

- Take off-shell $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet $L^{++}(\zeta, u)$ instead of the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ one $q^{+a}(\zeta, u)$ as the dynamical (co-ordinate) multiplet and still the gauged $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet $v^{+}(\zeta, u), \bar{v}^{+}(\zeta, u)$ to represent semi-dynamical spin degrees of freedom. This gives rise to $\mathcal{N} = 4, 3D$ SQM with coupling to non-abelian $3D$ gauge background (E.I. & M.Konyushikhin, 2010).
- The total superfield action is

$$\int du\, dt\, d^4\theta\, R_{\text{kin}}(L^{++}, L^{+-}, L^{--}, u) - \frac{ik}{2} \int du\, d\zeta^{(-2)}\, V^{++}$$

$$-\frac{1}{2} \int du\, dt\, d\bar{\theta}^{+} d\theta^{+}\, K(L^{++}, u) v^{+} \bar{v}^{+} \equiv S_1 + S_2 + S_3\,.$$

The first two pieces are the kinetic sigma-model type term of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet and FI term of the gauge $\mathcal{N} = 4$ multiplet. The third term describes coupling of the co-ordinate multiplet to the external gauge background.

Component action

- Take for simplicity free kinetic term for L^{++}, $\sim L^{++}(D^{--})^2 L^{++}$. The component form of the bosonic part of S is

$$S \to \int dt \left[\frac{1}{2}\, \dot{\ell}_m^2 + \mathcal{A}_m^a T^a \dot{\ell}_m + i\bar{\varphi}^\alpha \left(\dot{\varphi}_\alpha + iB\varphi_\alpha \right) + kB + \frac{1}{8} F^2 + \frac{1}{2} F \left(U^a T^a \right) \right]$$

$$\mathcal{A}_m^a = -\varepsilon_{mna} \partial_n \ln h, \quad U^a = -\partial_a \ln h\,, \quad T^a = \frac{1}{2} \bar{\varphi}^\alpha \left(\sigma_a \right)_\alpha^{\ \beta} \varphi_\beta\,,$$

$$h(\ell) = \int du\, K \left(\ell^{\alpha\beta} u_\alpha^+ u_\beta^+, u_\gamma^\pm \right), \quad \Delta h = 0\,.$$

- The $3D$ gauge field $\mathcal{A}_m^a$ and potential U^a are particular solutions of the Bogomolny equations

$$\mathcal{F}_{mn}^a = \varepsilon_{mns} \nabla_s U^a\,,$$

with $\mathcal{F}_{mn}^a = \partial_m \mathcal{A}_n^a - \partial_n \mathcal{A}_m^a + \varepsilon^{abc} \mathcal{A}_m^b \mathcal{A}_n^c\,, \qquad \nabla_m U^a = \partial_m U^a + \varepsilon^{abc} \mathcal{A}_m^b U^c\,.$

Quantization and example

- Quantization follow the same line as in the $4D$ case:
  $[T^a, T^b] = i\varepsilon^{abc} T^c\,, \quad T^a T^a = \frac{k}{2}\left(\frac{k}{2}+1\right).$
- The Hamiltonian in the case with free kinetic term for ℓ_i is

  $$H = \frac{1}{2}\left(\hat{p}_i - \mathcal{A}_i\right)^2 + \frac{1}{2}U^2 + fermionic\;\; terms, \quad \text{with } U \equiv U^a T^a\,.$$

  A new feature is the appearance of the "induced" potential term $\sim U^a U^b T^a T^b$ which originates from elimination of the auxiliary field F.
- $SO(3)$ invariant example:

  $$h_{\mathrm{so}(3)}(\ell) = c_0 + c_1 \frac{1}{\sqrt{\ell^2}}\,, \;\rightarrow\; \mathcal{A}^a_m = \varepsilon_{mna}\frac{\ell_n}{\ell^2}\frac{c_1}{c_1 + c_0\sqrt{\ell^2}}, \quad U^a = \frac{\ell_a}{\ell^2}\frac{c_1}{c_1 + c_0\sqrt{\ell^2}}.$$

- In the limit $c_0 = 0$ - Wu-Yang monopole (Wu-Yang, 1975); the $\mathcal{N} = 4$ SQM for this case was earlier constructed by Bellucci, Krivonos & Sutulin, 0911.3257 [hep-th], in a different approach.

Summary and Outlook

- $\mathcal{N} = 4, d = 1$ harmonic superspace is a useful tool of constructing and analyzing SQM models with $\mathcal{N} = 4$ SUSY. It allows one to construct off-shell invariant actions, to establish interrelations between different multiplets, to reveal the relevant target geometries, etc.
- Off-shell $\mathcal{N} = 4$ SUSY couplings of the multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ to the external non-abelian gauge backgrounds can be constructed using the auxiliary spin $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet. The background should be self-dual and be described by the 't Hooft ansatz or its static $3D$ reduction. HSS is indispensable for setting up off-shell actions.
- For the time being, our off-shell superfield construction is limited to this ansatz and to the gauge group SU(2). It is still open question how to extend it to the SU(N) gauge group and to the general self-dual background, e.g., to the renowned ADHM one.
- Surprisingly, the on-shell actions, with all auxiliary fields being eliminated, admit a direct extension to SU(N) and general self-dual backgrounds (Konyushikhin & Smilga, 2009; Ivanov, Konyushikhin & Smilga, 2009; Ivanov & Konyushikhin, 2010). This is feasible only at cost of on-shell realization of $\mathcal{N} = 4$ SUSY. Is it still possible to reproduce these models from an off-shell superfield approach?

- *Some further lines of development*: (a) Extensions to higher $\mathcal{N}$ SUSY, e.g. $\mathcal{N} = 8$; (b) Making use of some other $\mathcal{N} = 4$ multiplets to represent the coordinate and/or spin variable sectors, e.g. nonlinear versions of the multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$, etc (in progress).

- *Possible applications*: (a) For the explicit calculation of the world-line superextensions of Wilson loops, with the evolution parameter along the loop as a “time”; (b) In superextensons of Landau problem and higher-dimensional quantum Hall effect; (c) In supersymmetric black holes; (d) Somewhere else...

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