Discrete evolution operator for *q*-deformed top and Faddeev's modular double

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Notion of Faddeev's modular double

Consider standard Heisenberg algebra (HA) generated by operators x, p

$$[x,\ p] = i\ .$$

Introduce the algebra T (quantum torus) with generators U, V

$$U = e^{i\alpha x}\ , \quad V = e^{i\beta p}\ ,$$

$(\alpha, \beta$ are parameters) with commutation relations

$$U\,V = q\,V\,U \qquad q = e^{-i\,\alpha\beta}\ .$$

Is the algebra T of quantum torus (in above realization) is "equivalent" (representation theories are identical) to the Heisenberg algebra?

The answer is NO!

Notion of Faddeev's modular double

Indeed, one can construct another "dual" algebra $\widetilde{T}$ of quantum torus

$$\tilde{U} = e^{i\tilde{\alpha}x}\ , \quad \tilde{V} = e^{i\tilde{\beta}p}\ .$$

$$\tilde{U}\,\tilde{V} = \tilde{q}\,\tilde{V}\,\tilde{U}\ , \quad \tilde{q} = e^{-i\,\tilde{\alpha}\tilde{\beta}}\ ,$$

with another parameters $\tilde{\alpha}, \tilde{\beta}$. Then, if $\boxed{\tilde{\alpha} = \frac{2\pi}{\beta}\ , \quad \tilde{\beta} = -\frac{2\pi}{\alpha}\ ,}$, the generators U, V of T commute with generators $\tilde{U}\,\tilde{V}$ of $\widetilde{T}$ and deformation parameters q and $\tilde{q}$ are related by modular transformation

$$q = e^{-i\,\alpha\beta} = e^{i2\pi\tau} \ \rightarrow\ \tilde{q} = e^{-i\,\tilde{\alpha}\tilde{\beta}} = e^{-\frac{i2\pi}{\tau}} \quad (\tau \ \rightarrow\ \tilde{\tau} = -\frac{1}{\tau})\ .$$

The double of algebras T and $\tilde{T}$ is called modular double.

The modular double of T and $\tilde{T}$ is "equivalent to HA"!

The notion of the modular double was introduced by L.D.Faddeev for the case of $U_q(sl(2))$ (1995-1999).

Notion of Faddeev's modular double

Let x be a coordinate and p be a momentum of a free particle. The evolution of this particle with evolution operator $\Theta(t) = \exp(\frac{i}{2}p^2\, t)$:

$$p \to \Theta(t)\, p\, \Theta(t)^{-1} = p, \quad x \to \Theta(t)\, x\, \Theta(t)^{-1} = x + p\, t.$$

For coordinates U, V of quantum torus T, we obtain the evolution

$$V \to \Theta(t)\, V\, \Theta(t)^{-1} = V\ , \quad U \to \Theta(t)\, U\, \Theta(t)^{-1} = U\, e^{i\alpha tp}\, e^{\frac{i\alpha^2 t}{2}}\ ,$$

and for special interval $t = \frac{\beta}{\alpha} = -\frac{\tilde{\beta}}{\tilde{\alpha}}$ we obtain discrete evolution on T

$$V \to \Theta(V)\, V\, \Theta(V)^{-1} = V\ , \quad U \to \Theta(V)\, U\, \Theta(V)^{-1} = U\, V\, q^{-\frac{1}{2}}\ ,$$

where we denote $\Theta(V) = \Theta(\frac{\beta}{\alpha})$. This leads to the equation on $\Theta(V)$

$$\Theta(V) = q^{\frac{1}{2}}\ \Theta(qV)\, V\ ,$$

which can be solved in terms of the Jacobi theta-function ($|q| < 1$)

$$\Theta(V) = \prod_{n=1}^{\infty}(1 + q^{n-1/2}V) \prod_{n=1}^{\infty}(1 + q^{n-1/2}V^{-1})\ .$$

Notion of Faddeev's modular double

How can one relate evolution operator $\Theta(t) = \exp(\frac{i}{2}p^2\, t)$ and evolution operators

$$\Theta(V,q) = \prod_{n=1}^{\infty}(1+q^{n-1/2}V)\prod_{n=1}^{\infty}(1+q^{n-1/2}V^{-1}) \ ,$$
$$\Theta(\tilde{V},\tilde{q}) = \prod_{n=1}^{\infty}(1+\tilde{q}^{n-1/2}\tilde{V})\prod_{n=1}^{\infty}(1+\tilde{q}^{n-1/2}\tilde{V}^{-1}) \ ,$$

which are "compact" evolution operators for dual quantum torus T and $\widetilde{T}$, respectively?
The answer is given by known identity for theta-functions

$$\exp\left(\frac{i}{2}p^2\,\frac{\beta}{\alpha}\right) \sim \frac{\Theta(V,q)}{\Theta(\tilde{V},\tilde{q})} \ .$$

Below we obtain these formulas in the context of a discrete evolution of the $SL_q(2)$- quantum top model introduced by Faddeev and Alekseev.

1. R-matrices

Let V be a finite dimensional **C**-linear space. For any operator $X \in \mathrm{End}(V \otimes V)$ and integers $i > 0$, $j > 0$ we denote

$$X_{i\,i+1} := I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(j-1)} \in \mathrm{End}(V^{\otimes(i+j)}),$$

where $I \in \mathrm{Aut}(V)$ is the identity operator.

Def 1. An operator $\hat{R} \in \mathrm{Aut}(V \otimes V)$ is called an R-*matrix* if

$$\hat{R}^{i_1 i_2}_{k_1 k_2}\, \hat{R}^{k_2 i_3}_{n_2 j_3}\, \hat{R}^{k_1 n_2}_{j_1 j_2} = \hat{R}^{i_2 i_3}_{k_2 k_3}\, \hat{R}^{i_1 k_2}_{j_1 n_2}\, \hat{R}^{n_2 k_3}_{j_2 j_3}.$$

These braid relations can be written in concise matrix form:

$$\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12} = \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23}$$

Def 2. An R-matrix $\hat{R}$ is called a Hecke type R-matrix if

$$(\hat{R} - q\,\mathbf{1})(\hat{R} + q^{-1}\,\mathbf{1}) = 0\,, \quad (\mathbf{1} = I \otimes I)\,.$$

1. R-matrices

Consider the set of antisymmetrizers $\mathcal{A}^{(k)}(q)$ which can be defined by recurrent relations:

$$\mathcal{A}^{(k+1)} = \frac{[k]_q}{[k+1]_q}\mathcal{A}^{(k)}\left(\frac{q^k}{[k]_q} - \hat{R}_k\right)\mathcal{A}^{(k)} \in \mathrm{End}(V^{\otimes(k+1)})\,.$$

Def 3. A Hecke type R-matrix $\hat{R}$ for q – generic is called $GL_q(n)$ type R-matrix if it satisfies

$$1.)\;\; \mathcal{A}^{(n+1)} = 0 \;\;\Leftrightarrow\;\; \mathcal{A}^{(n)}\Big(\frac{q^n}{[n]_q}I - \hat{R}_n\Big)\mathcal{A}^{(n)} = 0\,, \quad 2.)\;\; \mathrm{rk}(\mathcal{A}^{(n)}) = 1\,.$$

An example of $GL_q(n)$ type R-matrix is the standard Drinfeld-Jimbo's R-matrix

$$\hat{R}^{\circ} \;=\; \sum_{i,j=1}^{n} q^{\delta_{ij}}E_{ij}\otimes E_{ji} \,+\, (q-q^{-1})\sum_{i<j}E_{ii}\otimes E_{jj}\,,$$

where $(E_{ij})_{kl} := \delta_{ik}\delta_{jl}$ are $(n\times n)$ matrix units.

Def 4. $\hat{R}$ is called *skew invertible* if $\exists\, \Psi \in End(V^{\otimes 2})$ such that

$$\hat{R}^{i_1 m_2}_{j_1 k_2}\, \Psi^{k_2 i_3}_{m_2 j_3} = \Psi^{i_1 m_2}_{j_1 k_2}\, \hat{R}^{k_2 i_3}_{m_2 j_3} = \delta^{i_1}_{j_3}\, \delta^{i_3}_{j_1}\ .$$

With any skew invertible $\hat{R}$ we associate matrix $D \in \mathrm{End}(V)$:

$$D_1 = \mathrm{Tr}_{(2)} \Psi_{12}\ ,$$

where $\mathrm{Tr}_{(i)}$ – trace in i-th space. Then, we define a quantum trace (q-traces) for any quantum matrix Y

$$Y \mapsto \mathrm{Tr}_{D}(Y) := \mathrm{Tr}(D\, Y)\,,$$

which possesses many remarkable properties, e.g.,

$$\mathrm{Tr}_{D(2)}(\hat{R}^{\varepsilon}_{12}\, Y_1\, \hat{R}^{-\varepsilon}_{12}) = I_1\, \mathrm{Tr}_{D}(Y)\ \ (\varepsilon = \pm 1)\,,$$

$$\mathrm{Tr}_{D(1,\dots,k)}\left(\left[\hat{R}_{i\,i+1},\ Y_{(1\dots k)}\right]\right) = 0\ \ (\forall\, 1 < i < k\,,\ \forall Y_{(1\dots k)})\,.$$

Quantized functions over matrix group (RTT algebra) (L.Faddeev,N.Reshetikhin,L.Takhtajan (1989)).

Let $\hat{R}$ be a skew invertible R-matrix. Consider an associative unital algebra generated by matrix components $\| T^i_j \|_{i,j=1}^{\dim V}$ which satisfy

$$\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12}\, .$$

The extension of this algebra by a set of components $\|(T^{-1})^i_j\|_{i,j=1}^{\dim V}$:

$$\sum_k T^i_k\, (T^{-1})^k_j = \sum_k (T^{-1})^i_k\, T^k_j = \delta^i_j\, 1\, ,$$

is a Hopf algebra with coproduct, counit and antipode mappings:

$$\Delta(T^i_j) = \sum_k T^i_k \otimes T^k_j\, , \qquad \epsilon(T^i_j) = \delta^i_j\, , \qquad S(T^i_j) = (T^{-1})^i_j\, .$$

This algebra is called an <u>*RTT algebra*</u> and denoted by $\mathcal{F}[\hat{R}]$.

Def 5. Let $\hat{R}$ be a skew invertible R-matrix. An associative unital algebra $\mathcal{L}[\hat{R}]$ with generators $\|L^i_j\|_{i,j=1}^{\dim V}$ satisfying relations

$$L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12} \;=\; \hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1\;,$$

is called a *reflection equation (RE) algebra*.

Consider REA $\mathcal{L}[\hat{R}]$ for Hecke type $\hat{R}$ and introduce elements ($a_0 = 1$)

$$a_i = \mathrm{Tr}_{D(1,\dots,i)}\left(\mathcal{A}^{(i)} L_{\overline{1}}\dots L_{\overline{i}}\right), \quad p_i = \mathrm{Tr}_{D}(L^i) \;\; (i \geq 1)$$

where $L_{\overline{1}} := L_1\,, \quad L_{\overline{k+1}} := \hat{R}_k\,L_{\overline{k}}\,\hat{R}_k^{-1}$. Elements p_i and a_i are central and called *power sums* and *elementary symmetric functions*, resp.

Prop. 1. Quantum Newton relations and *q*- Cayley-Hamilton identity

$$i_q\,a_i + (-1)^i \textstyle\sum_{j=0}^{i-1} (-q)^j a_j\,p_{i-j} = 0 \quad \forall\, i \geq 1\,,$$

$$\textstyle\sum_{j=0}^{n} (-q)^j\,a_j\,L^{n-j} \;=\; 0\,.$$

Prop. 2 *The set of* elementary symmetric functions $\{a_j,\, j=1,...,n\}$ *generate the whole center in REA* $\mathcal{L}[\hat{R}_{GL_q(n)}]$.

Def 5. A spectral extension of REA $\mathcal{L}[\hat{R}]$ for $GL_q(n)$ type $\hat{R}$-matrix is the extension of $\mathcal{L}[\hat{R}]$ by a set of invertible central elements μ_α $(\alpha = 1, \dots, n)$ such that

$$[\mu_\alpha,\, L^i_j] = 0$$

and

$$a_i = \sum_{1\le j_1<\dots<j_i\le n} \mu_{j_1}\mu_{j_2}\dots\mu_{j_i} \quad \forall\, i = 1, \dots, n.$$

It means that the Cayley-Hamilton identity can be written in factorized form

$$\sum_{j=0}^{n} (-q)^j\, a_j\, L^{n-j} \;=\prod_{\alpha=1}^{n}\left(L - q\mu_\alpha I\right) = 0\,.$$

Def 6. A *Heisenberg double (HD) algebra* of the *RTT* and RE algebras is an associative unital algebra generated by elements $T^i_j \in \mathcal{F}[\hat{R}]$ and $L^i_j \in \mathcal{L}[\hat{R}]$ subject to commutation relations

$$\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12}\ .$$

$$L_1\, \hat{R}_{12}\, L_1\, \hat{R}_{12} \ = \ \hat{R}_{12}\, L_1\, \hat{R}_{12}\, L_1\ ,$$

$$\gamma^2\, T_1\, L_2 \ = \ \hat{R}_{12}\, L_1\, \hat{R}_{12}\, T_1\, , \quad (\gamma \in \{\mathbf{C}\backslash 0\})\, .$$

This algebra is a quantization of the Poisson structure on $T^*(GL(n))$:

$$\{T^i_j,\ T^k_m\} = 0,\ \{\ell^i_j,\ \ell^k_m\} = 2(\delta^i_m \ell^k_j - \delta^k_j \ell^i_m),\ \{\ell^i_j,\ T^k_m\} = \delta^k_j T^i_m.$$

4. Heisenberg double of *RTT* and RE algebras

$$T_j^i \to T_j^i\,, \;\; L_j^i \to \delta_j^i + h\,\ell_j^i + \dots\,, \;\; R_{km}^{ij} \to \delta_m^i \delta_k^j + h\,\delta_k^i \delta_m^j + \dots\,.$$

HD algebra is interpreted as quantum group cotangent bundle, where *RTT* algebra is a base and RE algebra is a bundle.

For the spectral extension of HD we have additional commutators of T_j^i and L_j^i with spectral elements $\{\mu_\alpha\}$

$$[\mu_\alpha,\, L_j^i] = 0, \;\; [\mu_\alpha,\, T_j^i] = \dots$$

5. Discrete time evolution on quantum group cotangent bundle

Consider sequence of automorphisms on the HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$

$$\{T,\ L\} \xrightarrow{\theta^k} \{T(k),\ L(k)\}\,,\ \ \forall\, k = 0, 1, 2, \dots\,,$$

$$\begin{aligned} \hat{R}_{12}\, T_1(k)\, T_2(k) \ &= \ T_1(k)\, T_2(k) \hat{R}_{12} \\ \hat{R}_{12}\, L_1(k) \hat{R}_{12}\, L_1(k) \ &= \ L_1(k)\, \hat{R}_{12}\, L_1(k) \hat{R}_{12}\,, \\ \gamma^2\, T_1(k)\, L_2(k) \ &= \ \hat{R}_{12}\, L_1(k)\, \hat{R}_{12}\, T_1(k)\,. \end{aligned}$$

Here k is a discrete time. For any $\hat{R}$-matrix these automorphisms can be realized as (Faddeev–Alekseev discrete time evolution for the quantum top)

$$T(k) = L^k\, T\,, \quad L(k) = L\,.$$

5. Discrete time evolution for $SL_q(n)$ case

Consider the case when *RTT* algebra is $SL_q(n)$ quantum group. In this case we require

$$\det_q(T) = \mathrm{Tr}_{(1,\dots,n)}\left(\mathcal{A}^{(n)} T_1\, T_2 \cdots T_n\right) = 1\ .$$

Discrete time evolution must conserve this relation, i.e., we have $\det_q(L^k\, T) = 1$ $(\forall k > 0)$. This leads to the conditions

$$a_n = \mathrm{Tr}_{D(1,\dots,n)}\left(\mathcal{A}^{(n)} L_{\bar{1}}\, L_{\bar{2}} \cdots L_{\bar{n}}\right) = q^{-1}\ , \quad \gamma^n = q\ .$$

We will investigate the discrete evolution for HD of $SL_q(N)$ type. The key point is that $\exists$ the special evolution operator Θ:

$$T(k+1) = L\, T(k) = \Theta\, T(k)\, \Theta^{-1}\ , \quad L(k+1) = \Theta\, L(k)\, \Theta^{-1} = L\ .$$

For the case of "ribbon Hopf algebra" the Faddeev-Alekseev evolution is given by $\Theta =$ ribbon element.

6. Evolution operator Θ for $SL_q(n)$ case.

Thus, we have for the first shift $k = 1$:

$$L\,T = \Theta\,T\,\Theta^{-1}\,, \quad L = \Theta\,L\,\Theta^{-1}\,, \qquad (1)$$

and we assume $\Theta = \Theta(\mu_1, \dots, \mu_n)$, where $\prod_{\alpha=1}^{n} \mu_\alpha = q^{-1}$.

For the HD with $\hat{R}$-matrix of the $SL_q(n)$-type the evolution operator $\Theta(\mu_\alpha)$ is a solution of eqs. (1) which are written as

$$\Theta\big(\nabla^\alpha(\mu_\beta)\big) \; = q^{-1}\mu_\alpha^{-1}\,\Theta(\mu_\beta) \quad \forall\, \alpha = 1, \dots, n\,, \qquad (2)$$

where ∇^α are finite shift operators $\nabla^\alpha(\mu_\beta) \; := \; q^{2X_{\alpha\beta}}\,\mu_\beta$ and the matrix X is a Gram matrix

$$X_{\alpha\beta} \; = \; \langle \vec{e}_\alpha^{\,*}, \vec{e}_\beta^{\,*} \rangle \; = \; \delta_{\alpha\beta} \; - \; \frac{1}{n} \quad (\alpha, \beta = 1, \dots, n)\,,$$

for the set of vectors: $\vec{e}_\alpha^{\,*} = \frac{1}{n}\,(\underbrace{-1, \dots, -1}_{(\alpha-1)\ times}, n-1, -1, \dots, -1\,)$.

As a result we obtain (special solution):

Proposition. *In case $|q| < 1$ a solution is expressed via multidimensional theta-function*

$$\Theta^{(1)}(\mu_\alpha) = \theta(\vec{p}, \Omega) = \sum_{\vec{k} \in \mathbf{Z}^{n-1}} \exp\left\{\pi \mathrm{i}\, (\vec{k}, \Omega\, \vec{k}) + 2\pi \mathrm{i}\, (\vec{k}\,, \vec{p})\right\},$$

where τ is a modular parameter, Ω is $(n-1) \times (n-1)$ matrix of periods

$$q = \exp(2\pi \mathrm{i}\, \tau), \quad q^{1/n}\mu_\alpha = \exp(2\pi \mathrm{i}\, p_\alpha), \;\; \textstyle\sum_{\alpha=1}^{n} p_\alpha = 0,$$

$$\Omega_{\alpha\beta} = \tfrac{2\tau}{n} A^*_{\alpha\beta} = 2\tau\, (\delta_{\alpha\beta} - \tfrac{1}{n})\,,$$

Expression $\Theta^{(1)}(\mu_\alpha)$ converges either if $|q| < 1$, or if $q^m = 1$ (the series is truncated).

The $(n-1) \times (n-1)$ matrix $A^*_{\alpha\beta}$ is a Gram matrix of a lattice A^*_{n-1} dual to the root lattice $A_{n-1} = sl(n)$, since we have ${A^*}^{-1}_{\alpha\beta} = A_{\alpha\beta} = (\delta_{\alpha\beta} + 1)$ and $A_{\alpha\beta} = (e_\alpha, e_\beta)$, where vectors $e_\alpha = (\underbrace{0, \ldots, 0}_{(\alpha-1)\text{ times}}, 1, 0, \ldots, 0, -1)$ form the basis in the root space of $sl(n)$.

7. "Noncompact" solution for the evolution operator Θ

Proposition. *In case* $|q| \geq 1$ *one can find another solution:*

$$\Theta^{(2)}(p_\alpha) \;:=\; \exp\Big(-\frac{\pi\mathrm{i}}{2\tau}\sum_{\beta=1}^{n} p_\beta^2\Big)\,,$$

of the evolution equations.

Written in the independent variables $\vec{p} = \{p_1, \ldots, p_{n-1}\}$ it reads

$$\Theta^{(2)}(\vec{p}) \;=\; \exp\Big(-\frac{\pi\mathrm{i}}{\tau}\sum_{1\leq\alpha\leq\beta\leq n-1} p_\alpha p_\beta\Big) \;=\; \exp\Big\{-\pi\mathrm{i}\,(\vec{p},\,\Omega^{-1}\vec{p})\Big\}\,,$$

where the inverse matrix of periods is

$$\Omega^{-1}_{\alpha\beta} = \frac{1}{2\tau}\,(\delta_{\alpha\beta} + 1) = \frac{1}{2\tau}\,A_{\alpha\beta}\,,$$

and $A_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle$ is the Gram matrix for the root lattice A_{n-1}. Note that the logarithmic change of variables: $\log(\mu_\alpha)/(2\pi i) = p_\alpha - \tau/n$ which was rather superficial in case of $\Theta^{(1)}$, is inevitable for the derivation of $\Theta^{(2)}$.

Finally, we comment on relation between the two evolution operators $\Theta^{(1)} = \theta(\vec{p}, \Omega)$ and $\Theta^{(2)}$. The relation is based on the identity for multidimensional theta functions

$$\theta(\Omega^{-1}\vec{p}, -\Omega^{-1}) = \Big(\det(\Omega/\mathrm{i})\Big)^{\frac{1}{2}} \exp\Big\{\pi\mathrm{i}(\vec{p}, \Omega^{-1}\vec{p})\Big\}\, \theta(\vec{p}, \Omega)\,.$$

With our particular matrix of periods Ω we find

$$\Theta^{(2)}(\vec{p}) = \frac{1}{\sqrt{n}}\Big(\frac{2\tau}{\mathrm{i}}\Big)^{\frac{n-1}{2}} \frac{\theta(\vec{p}, \Omega)}{\theta(\Omega^{-1}\vec{p}, -\Omega^{-1})}\,.$$

Note that theta function $\theta(\Omega^{-1}\vec{p}, -\Omega^{-1})$ (in the denominator) commutes with the elements of HD (with $SL_q(n)$ $\hat{R}$-matrix) and can be thought as an evolution operator on a 'modular dual' quantum cotangent bundle associated to dual $\hat{R}$-matrix of $SL_{\tilde{q}}(n)$ type.

8. Example

In the $SL_q(2)$ case the evolution operator $\Theta^{(1)}$ becomes the Jacobi theta function (L.D. Faddeev (1995)):

$$\Theta^{(1)}(\mu_1) \;=\; \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}k(k+1)}\mu_1^k \;=\; \sum_{k\in\mathbb{Z}} \exp(\pi\mathrm{i}\,k^2\tau + 2\pi\mathrm{i}\,kz_1) \;=\; \theta_3(z_1;\,q)\,,$$

where $q = \exp(2\pi\mathrm{i}\,\tau)$, $\mu_1 = \exp(2\pi\mathrm{i}\,z_1)q^{-1/2}$. A multiplicative form for Θ is

$$\frac{1}{\eta(q)}\Theta^{(1)}(\mu_1) = \prod_{n=1}^{\infty}(1+q^n\mu_1)(1+q^{n-1}/\mu_1) = \prod_{n=1}^{\infty}(1+q^n\sigma_1+q^{2n-1})\,,$$

where $\eta(q) = \prod_{n=1}^{\infty}(1-q^n)$. For dual evolution operator we have

$$\widetilde{\Theta}^{(1)}(\mu_1) \;= \sum_{k\in\mathbb{Z}} \exp(-\frac{\pi\mathrm{i}}{\tau}k^2 + \frac{2\pi\mathrm{i}}{\tau}kz_1) = \sum_{k\in\mathbb{Z}} \tilde{q}^{\frac{1}{2}k(k+1)}\tilde{\mu}_1^k\,,$$

where $\tilde{q} = \exp(-\frac{2\pi\mathrm{i}}{\tau})$, $\tilde{\mu}_1 = \exp(\frac{2\pi\mathrm{i}}{\tau}z_1)\tilde{q}^{-1/2}$.

Summary

- What is a dual HD for the standard HD of $SL_q(n)$ type (which centralize each other)?
- Explicit expressions for evolution operator Θ in the case of B, C, D quantum groups. In these cases Gram matrices A and their dual $A^* = (A)^{-1}$ are such that B and C type evolution operators are dual to each other.
- 3D analogue of RE (tetrahedron RE) were proposed in A.P.Isaev and P.P.Kulish, Mod. Phys. Lett. **A12** (1997) 427 (hep-th/9702013). The analog of 3D *RTT* algebra is also known: $R_{123} T_1 T_2 T_3 = T_3 T_2 T_1 R_{123}$. What kind of cross-commutation relations are needed to describe discrete evolution in 3D case?