Invariants of the spherical sector in conformal mechanics

Tigran Hakobyan

Yerevan State University & Yerevan Physics Institute, Armenia

The talk is based on the results obtained in collaboration with
S. Krivonos, O. Lechtenfeld, A. Nersessian, A. Saghatelyan, V. Yeghikyan

International Workshop "Supersymmetry in Integrable Systems", Yerevan, August 24-28, 2010

The goal of the talk

The main goal of this talk is the investigation of the spherical (angular) part of the conformal mechanics. Considered as a separate Hamiltonian, it describes a mechanical system on the sphere, which we call briefly a "spherical mechanics".

In particular, we will

- Study the relation of the spherical part with other constants of motion of the general conformal mechanical system.
- Construct the constants of motion for spherical mechanics using the constants of motion in the underlying conformal mechanics.
- Construct the complete set of integrals in $2D$ spherical mechanics associated to the four-particle rational Calogero model.

The talk is based on the publications: *(1)* Hakobyan, Lechtenfeld, Nersessian, Saghatelyan, *Invariants of the spherical sector in conformal mechanics*, arXiv:1008.2912; *(2)* Hakobyan, Krivonos, Lechtenfeld, Nersessian, *Hidden symmetries of integrable conformal mechanical systems*, Phys. Lett. A **374**, 801 (2010); *(3)* Hakobyan, Nersessian, Yeghikyan, *Cuboctahedric Higgs oscillator from the Calogero model*, J. Phys. A **42**, 205206 (2009).

Motivation

- The relation of the spherical part of the Hamiltonian with other constants of motion has not been investigated properly so far.
- In particular, for the rational Calogero model, it is not a Liouville integral, but related to the additional integrals responsible for the superintegrability. It would be interesting to obtain the exact links among them.
- The study of the spherical mechanics has its own interest. It describes a particle motion on the sphere interacting with some potential fields, which can be considered as some multi-center high dimensional generalization of the spherical Higgs oscillator.
- The spherical mechanics of the Calogero model was used for the explanation of the non-equivalence of different quantizations of the Calogero model [Feher, Tsutsui, Fulop, 2005], as well as for the construction its superconformal generalizations [Bellucci, Krivonos, Sutulin, 2008].

Conformal Mechanics

The Hamiltonian of the conformal mechanics is [De Alfaro, Fubini, Furlan, 1974]

$$H = \frac{\vec{p}^{\,2}}{2} + V(\vec{r}), \qquad (\vec{r}\cdot\vec{\nabla})V(\vec{r}) = -2V(\vec{r}).$$

Together with the dilatation and special conformal transformation

$$D = \vec{p}\cdot\vec{r}, \qquad K = \frac{\vec{r}^{\,2}}{2},$$

it forms the conformal algebra $so(2,1)$

$$\{H, D\} = 2H, \qquad \{K, D\} = -2K, \qquad \{H, K\} = D.$$

Invariant representation:

$$J_{1,3} = H \pm K, \quad J_2 = D: \quad \{J_a, J_b\} = -2\varepsilon_{abc}J^c,$$

The indexes upper by $\gamma_{ab} = \mathrm{diag}(1,-1,-1)$.

The spherical part $\mathcal{I}$: spherical mechanics

The Casimir of $so(2,1)$ is a constant of motion quadratic on momenta:

$$\mathcal{I} = 4KH - D^2 = \sum_a J_a J^a, \qquad \{H, \mathcal{I}\} = 0.$$

In any spherical coordinates with the radial coordinates $r = \sqrt{2K}$ and $p_r = \frac{D}{\sqrt{2K}}$, we have:

$$\{p_r, r\} = 1, \qquad D = rp_r, \qquad K = \frac{r^2}{2}, \qquad H = \frac{p_r^2}{2} + \frac{\mathcal{I}}{2r^2},$$

where the Casimir element $\mathcal{I} = \mathcal{I}(u)$ depends on the angular coordinates $u = \{p_{\phi_\alpha}, \phi_\alpha\}$ only.

The spherical mechanics given by the Hamiltonian $\mathcal{I}(u)$ describes a particle on $(N-1)$-dimensional sphere moving in the presence of the external potential.

Notation: Denote by $\hat{F}$ the associated Hamiltonian vector field: $\hat{F}G := \{F, G\}$. In particular, $\hat{\mathcal{I}} = 4H\hat{K} + 4K\hat{H} - 2D\hat{D}$.

Integrals of conformal mechanics

Suppose, we have an integral of motion in the conformal mechanics. Without loss of generality, consider the integrals with a certain conformal spin:

$$\hat{H} I_s = \{H, I_s\} = 0, \qquad \hat{D} I_s = \{D, I_s\} = -2s\, I_s .$$

The conservation condition is equivalent to the equation

$$(\hat{\mathcal{I}} - \hat{M})\, I_s(p_r, r, u) = 0, \qquad \hat{M} = 2\hat{S}_- - 2\mathcal{I}\hat{S}_+, \tag{1}$$

where the $1D$ vector fields

$$\hat{S}_+ = \frac{1}{r}\frac{\partial}{\partial p_r}, \qquad \hat{S}_- = -p_r r^2 \frac{\partial}{\partial r}, \qquad \hat{S}_z = -\frac{1}{2}\left(r\frac{\partial}{\partial r} + p_r \frac{\partial}{\partial p_r} \right).$$

form $so(3)$ algebra, Decompose I_s in terms of spherical and radial functions

$$I_s(p_r, r, u) = \sum_{m=-s}^{s} f_{s,m}(u)\; R_{s,m}(p_r, r), \quad R_{s,m}(p_r, r) = \sqrt{\binom{2s}{s+m}}\, \frac{p_r^{s-m}}{r^{s+m}}.$$

The radial functions $R_{s,m}$ form standard spin s-representation of the $so(3)$ algebra $\hat{S}_\pm,\ \hat{S}_z$.

Equations of motion for the coefficients $f_{s,m}$ in spherical mechanics

From (1), the equations of motion of the $2s+1$ angular coefficients $f_{s,m}(u)$ in the spherical mechanics $\mathcal{I}$ is given by linear equations

$$\frac{df_{s,m}}{dt_{\mathcal{I}}} = \hat{\mathcal{I}} f_{s,m} = \sum_{m'=-s}^{s} M_{mm'} f_{s,m'} = 2\sqrt{(s-m)(s+m+1)} f_{s,m+1} - 2\mathcal{I}\sqrt{(s-m+1)(s+m)} f_{s,m-1}. \tag{2}$$

The related $(2s+1)$th-order linear homogeneous differential equation is

$$\mathrm{Det}(\hat{\mathcal{I}} - M) f_{s,m} = 0. \tag{3}$$

As a consequence, from the set of derivative integrals $I_s^{(k)} := \hat{\mathcal{I}}^k I_s$ obtained from I_s,

$$I_s \xrightarrow{\hat{\mathcal{I}}} I_s^{(1)} \xrightarrow{\hat{\mathcal{I}}} I_s^{(2)} \xrightarrow{\hat{\mathcal{I}}} \dots \xrightarrow{\hat{\mathcal{I}}} I_s^{(k)} \xrightarrow{\hat{\mathcal{I}}} \dots$$

the first $(2s+1)$ integrals *at most* may be functionally independent.

Properties of the spherical coefficients

- The map $I_s \longrightarrow f_{s,-s}$ (the contraction $r \longrightarrow \infty$) is a Poisson algebra homomorphism ($I_s = f_{s,-s}p_r^{2s} + \sqrt{2s}f_{s,-s+1}\frac{p_r^{2s-1}}{r} + \ldots$):

$$I_{s_1}I_{s_2} \longrightarrow f_{s_1,-s_1}f_{s_2,-s_2}, \qquad \{I_{s_1}, I_{s_2}\} \longrightarrow \{f_{s_1,-s_1}, f_{s_2,-s_2}\}.$$

- The integral I_s is completely determined by its leading coefficient on p_r, the others are defined recursively using the equation of motion (2):

$$f_{s,-s+1} = \frac{1}{2\sqrt{2}}\hat{\mathcal{I}}f_{s,-s}, \quad f_{s,-s+2} = \frac{1}{\sqrt{s(2s-1)}}\left(\frac{1}{8}\hat{\mathcal{I}}^2 + s\mathcal{I}\right)f_{s,-s}, \quad \ldots$$

- From two integral I_{s_1} and I_{s_2}, the "new" integrals can be constructed using the Clebsch-Gordan coefficients:

$$I'_S = \sum_m f'_{S,m}R_{S,m}, \quad f'_{S,m} = \sum_{m_1+m_2=m} C^{Sm}_{s_1m_1,s_2m_2}f_{s_1,m_1}f_{s_2,m_2},$$

where $|s_1 - s_2| \leq S \leq s_1 + s_2$. In particular, $I'_{s_1+s_2} = I_{s_1}I_{s_2}$.

The solutions to the equations of motion for $f_{s,m}$

The formal solution is provided by the diagonalization of the matrix M,

$$\hat{M} = 4\sqrt{-\mathcal{I}}\;\hat{U}\,\hat{S}_z\,\hat{U}^{-1}, \qquad \hat{U} = (-\mathcal{I})^{\frac{1}{2}\hat{S}_z}\,e^{-\frac{i\pi}{2}\hat{S}_y}$$

The eigenfunctions of the operator $\hat{M}$ are

$$\widetilde{R}_{s,m} = \sum_{m'} U_{m'm} R_{s,m'}, \quad U_{m'm} = d^s_{m'm}(\pi/2)(-\mathcal{I})^{\frac{m'}{2}}, \quad \hat{M}\widetilde{R}_{s,m} = m\widetilde{R}_{s,m},$$

where $d^s_{m'm}(\beta)$ is the Wigner's small d-matrix, which describes the rotation around the y axis in the spin-s representation.
In terms of the rotated radial functions, the expression of the integral becomes

$$I_s(p_r, r, u) = \sum_{m=-s}^{s} \widetilde{f}_{s,m}(u)\widetilde{R}_{s,m}(p_r, r, \mathcal{I}(u)),$$

$$\widetilde{f}_{s,m} = \sum_{m'} U^{-1}_{mm'} f_{s,m'} = \sum_{m'} (-\mathcal{I})^{-\frac{m'}{2}} d^s_{m'm}(\pi/2) f_{s,m'}.$$

The solutions to the equations of motion and integrals of $\mathcal{I}$

The rotated coefficients are (in general, complex) eigenfunctions of $\mathcal{I}$ and oscillate in time with the frequency $\omega_m = 4m\sqrt{\mathcal{I}}$:

$$\hat{\mathcal{I}}\widetilde{f}_{s,m}(u) = i\omega_m \widetilde{f}_{s,m}(u), \qquad \widetilde{f}_{s,m}(t) = e^{i\omega_m(t-t_0)}\widetilde{f}_{s,m}(t_0),$$

Using the eigenvalues of M, we obtain

$$\mathrm{Det}(\hat{\mathcal{I}} - M) = \prod_{m=-s}^{s}(\hat{\mathcal{I}} - 4m\sqrt{-\mathcal{I}}) =: \begin{cases} \hat{\mathcal{I}}\hat{\Delta}_s & \text{for } s \in \mathbb{Z}, \\ \hat{\Delta}_s & \text{for } s \in \mathbb{Z}+\frac{1}{2}, \end{cases}$$

$$\hat{\Delta}_s = \prod_{0<m\leq s}(\hat{\mathcal{I}}^2 + 16m^2\mathcal{I}).$$

So, the exact form of the equations (3) is

$$\hat{\Delta}_s f_{s,m} = 0 \quad \text{for } s = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, \quad \text{and} \quad \hat{\mathcal{I}}\hat{\Delta}_s f_{s,m} = 0 \quad \text{for } s = 1, 2, \ldots.$$

This means that for integer spins, $\hat{\Delta}_s f_{s,m}$ is an integral of motion (if nonzero) of the spherical mechanics $\mathcal{I}$.

Integrals of motion of the spherical mechanics linear in $f_{s,m}$

An integral I_s of the conformal mechanics H can be used to construct integrals of motion of the related spherical mechanics $\mathcal{I}$. They have the simplest form while expressed in terms of the oscillated complex functions $\widetilde{f}_{s,m}(u)$. We consider linear and bilinear expressions mainly.

Integral of $\mathcal{I}$ linear in $f_{s,m}$: integer spins only

For integer s, the $m = 0$ eigenfunction is an integral of $\mathcal{I}$. Using the explicit expression for $d^s_{mm'}(\pi/2)$, the modified integral (without the fractional powers of $\mathcal{I}$) is calculated

$$\mathcal{J}_s(u) = \mathcal{I}(u)^{\frac{s}{2}}\widetilde{f}_{s,0}(u) = \sum_{\ell=0}^{s} \frac{(2\ell-1)!!(2s-2\ell-1)!!}{\sqrt{(2s)!}} \mathcal{I}(u)^{\ell} f_{s,2\ell-s}(u).$$

Two constructed integrals are similar. The inverse transformation $f_{sm} = \sum_{m'} U_{mm'}\widetilde{f}_{sm'}$ gets

$$\hat{\Delta}_s f_{s,m} = U_{m0}\hat{\Delta}_s\widetilde{f}_{s,0} = \delta_{s-m,2\mathbb{Z}}\, c_{sm}\, \mathcal{I}^{\frac{s+m}{2}} \mathcal{J}_s,$$

c_{sm} is a numerical coefficient.

Integrals bilinear in $f_{s,m}$

Other constants of motion can be built also by bilinear combinations of $f_{s,m}$.

The bilinear integrals, $0 < m \leq s$

The following observables are well-defined (real, without fractional powers of $\mathcal{I}$) constants of motion of $\mathcal{I}$:

$$\mathcal{J}_s^m = (-\mathcal{I})^s \tilde{f}_{s,m} \tilde{f}_{s,-m}$$
$$= \sum_{m',m''} \delta_{m''-m',2\mathbb{Z}} (-1)^{2s+\frac{m''-m'}{2}} d^s_{m''m}(\pi/2) d^s_{m'm}(\pi/2)\, \mathcal{I}^{s-\frac{m'+m''}{2}} f_{s,m'} f_{s,m''}$$

One can choose the total spin "quantum" number S instead of m to represent this set of integrals:

$$\mathcal{F}_s^S(u) = \sum_m C^{S,0}_{sm,s-m} \mathcal{J}_m^s(u).$$

Unwanted fractional powers of $\mathcal{I}$ cancel in the expression.

Other integrals of $\mathcal{I}$

- The construction straightforwardly generalizes also to multilinear forms composed from the angular functions:

$$\mathcal{J}^{m_1\ldots m_k}_{s_1\ldots s_k}(u) = \mathcal{I}(u)^{\frac{1}{2}\sum_\ell s_\ell} \prod_{\ell=1}^{k} \widetilde{f}_{s_\ell,m_\ell}(u) \qquad \sum_{\ell=1}^{k} m_\ell = 0,$$

- The factors can also be taken into account like $\widetilde{f}_{s_1,m}(u)/\widetilde{f}_{s_2,m}(u)$.

A single integral I_s of the conformal mechanics H gives rise to a number of integrals of the spherical mechanics $\mathcal{I}$.

- The derived integrals are not independent, in general.
- If two integrals of the conformal mechanics are in involution, then the integrals of the spherical mechanics retrieved from them are not in involution. So, if H integrable with the Liouville integral set $\{I_s\}$, the related integrals of $\mathcal{I}$ do not define the set of Liouville integrals.

Calogero model

A particular case of the conformal mechanics is the N-particle rational Calogero model [Calogero, 1969,1971] $$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i<j} \frac{g^2}{(x_i - x_j)^2}$$

- Is integrable with N Liouville constants of motion given in terms of a Lax matrix [Moser, 1975]

$$I_s = \operatorname{Tr} L^{2s}, \quad s = \tfrac{1}{2}, 1, \ldots, \tfrac{N}{2}, \qquad L_{jk} = \delta_{jk} p_k + (1-\delta_{jk})\frac{ig}{x_j - x_k},$$

$$I_{\frac{1}{2}} = \sum_i p_i, \qquad I_1 = H$$

- Maximally superintegrable: has additional constants of motion ($N-1$ of them are independent) [Wojciechowski, 1973]

$$F_{s_1 s_2} = 2s_1 I_{s_1} \hat{K} I_{s_2} - 2s_2 I_{s_2} \hat{K} I_{s_1}.$$

The action of $\hat{\mathcal{I}}$ on the Liouville integrals produces the additional integrals:

$$I_s^{(1)} = \hat{\mathcal{I}} I_s = 2F_{1s}.$$

The spherical mechanics $\mathcal{I}$ as an extended Higgs oscillator.

The angular part $\mathcal{I}$ describes a particle motion on the sphere.

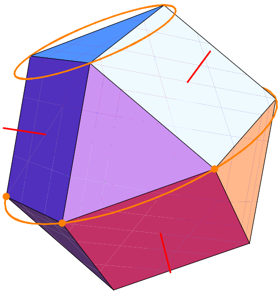
$$H = \frac{\vec{p}^2}{2} + \sum_{\vec{\alpha}\in\Delta_+} \frac{g^2}{(\vec{\alpha}\cdot\vec{x})^2} = \frac{p_r^2}{2} + \frac{\mathcal{I}(p_{\varphi_a},\varphi_a)}{2r^2},$$

$$\mathcal{I}(p_{\varphi_a},\varphi_a) = K_{\text{sph}}(p_{\varphi_a},\varphi_a) + \sum_{\alpha=1}^{N(N-1)/2} \frac{2g^2}{\cos^2\theta_\alpha}, \quad \{p_{\varphi_a},\varphi_b\} = \delta_{ab}$$

- $\Delta_+ = \{\epsilon_i - \epsilon_j, i > j\}$ is the set of positive roots of A_N [Olshanetsky, Perelomov],
- $\cos\theta_\alpha = \vec{\alpha}\cdot\vec{x}$, where $\theta_\alpha = \theta_\alpha(\varphi_a)$,
- $\frac{1}{2}\omega^2 r_0^2 \tan^2\theta$ is the potential of the Higgs oscillator on the sphere [Higgs, 1979].
- The center of mass of H can be reduced applying the orthogonal Jacobi transformations [Ioffe, Neelov, 2002].

The spherical part can be regarded as an integrable $\frac{1}{2}N(N-1)$ center high-dimensional extension of the Higgs oscillator with the frequency $\omega = 2g$.

The spherical mechanics of the four particle Calogero model with reduced center of mass



- The roots of A_3 Lie algebra are associated with the vertexes of cuboctahedron.
- The root system Δ has octahedral symmetry $O_h \equiv S_4 \otimes Z_2$ of order 48.

Three particle Calogero system: the algebra of integrals of motion

The spherical mechanics of $N = 3$ system with the reduced center-of-mass is $1D$ three-center Higgs oscillator [Jacobi]:

$$H = \frac{p_r^2}{2} + \frac{\mathcal{I}}{2r^2} \qquad \mathcal{I} = p_\varphi^2 + \frac{9g^2}{\cos^2 3\varphi},$$

The $2D$ conformal mechanics H has two $s = \frac{3}{2}$ integrals

$$I_{\frac{3}{2}} = \left(p_r^2 - \frac{6\mathcal{I}}{r^2}\right) p_r \sin 3\varphi + \left(3p_r^2 - \frac{2\mathcal{I}}{r^2}\right) \frac{p_\varphi \cos 3\varphi}{r},$$

$$I_{\frac{3}{2}}^{(1)} = \hat{\mathcal{I}} I_{\frac{3}{2}} = \left(p_r^2 - \frac{6\mathcal{I}}{r^2}\right) 6p_r p_\varphi \cos 3\varphi - \left(3p_r^2 - \frac{2\mathcal{I}}{r^2}\right) \frac{12\mathcal{I} \sin 3\varphi}{r}.$$

The nontrivial brackets (including $\mathcal{I}$) are:

$$\{\mathcal{I}, I_{\frac{3}{2}}\} = I_{\frac{3}{2}}^{(1)}, \qquad \{\mathcal{I}, I_{\frac{3}{2}}^{(1)}\} = -18\mathcal{I} I_{\frac{3}{2}}, \qquad \{I_{\frac{3}{2}}^{(1)}, I_{\frac{3}{2}}\} = 18(8H^3 - I_{\frac{3}{2}}^2).$$

Of course, only 3 integrals are independent:

$$\left(I_{\frac{3}{2}}^{(1)}\right)^2 + 36\mathcal{I} I_{\frac{3}{2}}^2 = 8H^3(\mathcal{I} - 9g^2).$$

4 particle spherical mechanics: the Hamiltonian $\mathcal{I}$

The transformation (Jacobi transformation + reflection)

$$y_0 = \tfrac{1}{2}(x_1+x_2+x_3+x_4), \qquad y_1 = \tfrac{1}{2}(x_1+x_2-x_3-x_4),$$
$$y_2 = \tfrac{1}{2}(x_1-x_2+x_3-x_4), \qquad y_3 = \tfrac{1}{2}(x_1-x_2-x_3+x_4)$$

decouples the center-of-mass coordinate y_0 and momentum of $N = 4$ Calogero model. After reducing them and passing to the spherical coordinates, the spherical Hamiltonian takes the form

$$\mathcal{I}(p_\theta, p_\varphi, \theta, \varphi) = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} + \frac{2g^2}{\sin^2\theta} \times \sum_{\pm} \left[\frac{1}{(\cos\varphi \pm \sin\varphi)^2} + \frac{1}{(\cot\theta \pm \sin\varphi)^2} + \frac{1}{(\cot\theta \pm \cos\varphi)^2} \right],$$

with the spherical symplectic structure $\omega_0 = dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi$.
It has 3 independent integrals (is max. superintegrable). As for general conformal mechanics, try to extract them from the integrals of underlying system $H = \frac{1}{2}(p_r^2 + \mathcal{I}/r^2)$.

4 particle Calogero: the algebra of integrals

The conformal Hamiltonian has two Liouville constants of motion $I_s = \mathrm{Tr} L^{2s}$ of conformal spin $s = \frac{3}{2}, 2$. Their leading-term coefficients $f_s := f_{s,-s}$ in the decomposition $I_s = \sum_m f_{s,m} R_{s,m}$ can be calculates easily:

$$f_{\frac{3}{2}}(\theta,\varphi) = \tfrac{3}{2}\cos\theta\sin^2\theta\sin 2\varphi, \qquad f_2(\theta,\varphi) = \tfrac{1}{4}\left(\sin^2 2\theta + \sin^4\theta\,\sin^2 2\varphi\right).$$

The Liouville integrals are supplemented by the two related Wojciechowski integrals $I_s^{(1)} = \hat{\mathcal{I}} I_s$ with leading-term coefficients linear in momenta: $g_s = \hat{\mathcal{I}} f_s$.

$$\{f_{\frac{3}{2}}, g_{\frac{3}{2}}\} = 18\,(f_{\frac{3}{2}}^2 - f_2), \quad \{f_2, g_2\} = 8\,(4f_2^2 - \tfrac{1}{3} f_{\frac{3}{2}}^2 - f_2), \quad \{f_{\frac{3}{2}}, f_2\} = 0,$$

$$\{f_{\frac{3}{2}}, g_2\} = \{f_2, g_{\frac{3}{2}}\} = 8 f_{\frac{3}{2}}(3f_2 - 1), \quad \{g_{\frac{3}{2}}, g_2\} = 4\,(2 g_{\frac{3}{2}} f_2 - 3 f_{\frac{3}{2}} g_2).$$

Since the map $I_s \mapsto f_s$ is a Poisson algebra homomorphism, we immediately get the nontrivial brackets for the set of 5 integrals H, I_s, $I_s^{(1)}$:

$$\{I_{\frac{3}{2}}, I_{\frac{3}{2}}^{(1)}\} = 18\,(I_{\frac{3}{2}}^2 - 2I_2 H), \quad \{I_2, I_2^{(1)}\} = 8\,(4I_2^2 - \tfrac{2}{3} I_{\frac{3}{2}}^2 H - 4 I_2 H^2),$$

$$\{I_{\frac{3}{2}}, I_2^{(1)}\} = \{I_2, I_{\frac{3}{2}}^{(1)}\} = 8 I_{\frac{3}{2}}(3I_2 - 4H^2), \quad \{I_{\frac{3}{2}}^{(1)}, I_2^{(1)}\} = 4\,(2 I_{\frac{3}{2}}^{(1)} I_2 - 3 I_{\frac{3}{2}} I_2^{(1)}).$$

This is a particular realization of the quadratic algebra related to the Hamiltonian [Kuznetzov, 1995]

The integrals $\mathcal{J}_s$ of spherical Hamiltonian $\mathcal{I}$

Using the leading angular coefficient f_2, the "linear" integral of $\mathcal{I}$ associated with I_2 is derived from its general form:

$$\mathcal{J}_2 = -\frac{1}{\sqrt{6}}\left(\frac{1}{256}\hat{\mathcal{I}}^4 + \frac{5}{16}\mathcal{I}\hat{\mathcal{I}}^2 + 4\mathcal{I}^2\right) f_2.$$

Its explicit expression is complicated:

$$\mathcal{J}_2 = \frac{1}{\sqrt{6}}\Bigg[\frac{1}{16}(3\cos 4\varphi - 11)\, p_\theta^4 - \frac{3}{4}\cot\theta\sin 4\varphi\ p_\theta^3 p_\varphi + \frac{3}{4}\cot^3\theta\sin 4\varphi\ p_\theta p_\varphi^3$$
$$-\left(\frac{11+9\cos 4\varphi}{8\sin^2\theta} + \frac{9}{4}\sin^2 2\varphi\right) p_\theta^2 p_\varphi^2 + \frac{3\cos^4\theta\cos 4\varphi + 21\sin^4\theta - 18\sin^2\theta - 11}{16\sin^4\theta}\, p_\varphi^4\Bigg]$$
$$+ g^2 K_1(\theta,\varphi)\, p_\theta^2 + g^2 K_2(\theta,\varphi)\, p_\theta p_\varphi + g^2 K_3(\theta,\varphi)\, p_\varphi^2 + g^4 K_4(\theta,\varphi),$$

The system of equations (2) can be applied in order to express the coefficients $f_{\frac{3}{2},m}$ in terms of $f_{\frac{3}{2}}$:

$$f_{\frac{3}{2},-\frac{1}{2}} = \frac{1}{2\sqrt{3}}\hat{\mathcal{I}} f_{\frac{3}{2}}, \quad f_{\frac{3}{2},\frac{1}{2}} = \left(\frac{1}{8\sqrt{3}}\hat{\mathcal{I}}^2 + \frac{\sqrt{3}}{2}\mathcal{I}\right) f_{\frac{3}{2}}, \quad f_{\frac{3}{2},\frac{3}{2}} = \left(\frac{1}{48}\hat{\mathcal{I}}^2 + \frac{7}{12}\mathcal{I}\right)\hat{\mathcal{I}} f_{\frac{3}{2}}.$$

The integrals $\mathcal{J}_s^m$ of spherical Hamiltonian $\mathcal{I}$

Then, using one obtains the spherical constants of motion associated with $I_{\frac{3}{2}}$, namely $\mathcal{J}_{\frac{3}{2}}^{\frac{3}{2}}$ and $\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}}$. Their explicit expressions are rather lengthy:

$$\begin{aligned}
\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}} = & -\tfrac{3}{32}\sin^2 2\varphi\, p_\theta^6 - \tfrac{3}{16}\cot\theta\sin 4\varphi\, p_\theta^5 p_\varphi - \tfrac{3}{128}\frac{6\cos^2\theta + (13-3\cos 2\theta)\cos 4\varphi}{\sin^2\theta} p_\theta^4 p_\varphi^2 \\
& -\tfrac{3}{128}\frac{22\sin^4\theta - (43-53\cos 2\theta)\cos 4\varphi\cos^2\theta + 6\cos 2\theta}{\sin^4\theta} p_\theta^2 p_\varphi^4 \\
& -\tfrac{3}{128}\frac{\cos^2\theta\left((5+11\cos 4\varphi)\sin^2\theta + (2-9\cos 2\theta\sin^2\theta)(1-\cos 4\varphi)\right)}{\sin^6\theta} p_\varphi^6 \\
& +\tfrac{3}{2}\cot\theta\sin 4\varphi\, p_\theta^3 p_\varphi^3 - \tfrac{3}{32}\frac{(7-9\cos 2\theta)\cos^3\theta\sin 4\varphi}{\sin^5\theta} p_\theta p_\varphi^5 + \ \ldots,
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{\frac{3}{2}}^{\frac{3}{2}} = & -\tfrac{9}{32}\sin^2 2\varphi\, p_\theta^6 - \tfrac{9}{16}\cot\theta\sin 4\varphi\, p_\theta^5 p_\varphi - \tfrac{9}{64}\left(\tfrac{5\cos 4\varphi + 3}{\sin^2\theta} + 10\sin^2 2\varphi\right) p_\theta^4 p_\varphi^2 \\
& -\tfrac{9}{64\sin^4\theta}\left(5\cos^4\theta\cos 4\varphi + 10\sin^2\theta - 5\sin^4\theta + 3\right) p_\theta^2 p_\varphi^4 + \tfrac{9}{16}\cot^5\theta\sin 4\varphi\, p_\theta p_\varphi^5 \\
& +\tfrac{9\cos^2\theta}{64\sin^6\theta}\left(\cos^4\theta\cos 4\varphi - 6\sin^2\theta - \sin^4\theta - 1\right) p_\varphi^6 \ + \ \ldots.
\end{aligned}$$

The algebraic relation among the four integrals of $\mathcal{I}$

- Clearly, $\mathcal{I}$, $\mathcal{J}_2$, $\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}}$ and $\mathcal{J}_{\frac{3}{2}}^{\frac{3}{2}}$ cannot be functionally independent, since the spherical mechanics is 2 dimensional.
- Using `Mathematica`, one can verify the following algebraic relation,

$$\mathcal{J}_{\frac{3}{2}}^{\frac{3}{2}} = \frac{1}{3}\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}} + 2\sqrt{\frac{2}{3}}\mathcal{J}_2\mathcal{I} + \frac{1}{3}\mathcal{I}^3 + 4g^2\mathcal{I}^2.$$

- This is the only relation among the four constants of motion, since it can be verified also that $\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}}$ and $\mathcal{J}_{\frac{3}{2}}^{\frac{3}{2}}$ are not in involution with $\mathcal{J}_2$. Even their free-particle parts ($g{=}0$ projects to the terms of highest order in the momenta) do not commute as is easy to verify.

Hence, $\mathcal{J}_2$, $\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}}$ together with $\mathcal{I}$ form the complete set of functionally independent constants of motion for the $2D$ spherical mechanics $\mathcal{I}$ associated with 4-particle Calogero model with excluded center of mass. This confirms the superintegrability of that system.

Summary and discussion

A general approach to the constants of motion for conformal mechanics, based on $so(3)$ representation theory is proposed.

- Constants of motion for the spherical mechanics are constructed from a constant of motion for the initial conformal system.
- We have illustrated the effectiveness of our method on the example of the rational A_3 Calogero model and its spherical mechanics. For the latter we have constructed a complete set of functionally independent constants of motion, proving its intuitively obvious superintegrability.
- Unfortunately, our approach does not allow one to select a commuting subset of constants of motion for the spherical mechanics (if integrable), as well as to reveal the functionally independent constants of motions.
- Another task is to construct the complete set of integrals for the spherical mechanics associated with N-particle Calogero system.