Defects in the Liouville field theory

Sarkissian Gor

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Yerevan, Armenia

Defects: General overview

Defects in two-dimensional quantum field theory are oriented lines separating different quantum field theories. The notion of the defects is very rich and defects appear in the numerous different topics, like condensed matter, string theory, algebraic topology, Langland theory, boundary conformal field theory, D-branes, $N = 2$ 4D gauge theories

Example 1 : Lagrangian approach

$$S = \int_{\Sigma_1} L_1 + \int_{\Sigma_2} L_2 + \int_{\partial\Sigma_1} L^{\text{def}} \qquad (1)$$

where $\partial\Sigma_1 = -\partial\Sigma_2$

Example 2: WZW model

$$g_1 g_2^{-1}|_{\text{defect}} = C_G^\mu, \qquad C_G^\mu = \beta e^{2i\pi\mu/k} \beta^{-1}, \qquad \beta \in G \tag{2}$$

where $\mu \equiv \boldsymbol{\mu} \cdot \boldsymbol{H}$ is a highest weight representation integrable at level k, taking value in the Cartan subalgebra.

Let us briefly comment about the role of defects in the mentioned topics.

- Condensed matter. In condensed matter defects play important role in the consideration of impurities , quantum wires, quantum Hall effects, renormalization group flow.

- String Theory. Defects appear as domain wall in String theory in the context of Ads/CFT correspondence

- Boundary conformal field theory and D-branes. Topological defects can be fused with boundary states producing new boundary states. Hence, defects can be used as efficient tool for producing new boundary states. Given that in String theory boundary states can be realized as D-branes in target space, defects can be associated to the various operations of the D-branes transformations, like T-dualities and monodromies.

- Algebraic topology. It is established by now that D-branes are classified by K-theory. Therefore topological defects can be associated to topological operations in K-theory. It is conjectured that defects can be interpreted as kernel of the Fourier-Mukai transform.

- Langland program. Topological defects also play an important role in the recently established connection between geometric Langland program and dimensionally reduced topologically twisted $N = 4$ four-dimensional super Yang-Mills theory.

- $N = 2$ 4D gauge theory. Expectation values of the Wilson- t' Hooft operators in a class of $N = 2$ theories are related to correlation functions in the presence of defects of the Liouville CFT.

Topological defects in RCFT

Maximally-symmetric topological defect lines are defined by the conditions:

$$T^{(1)} = T^{(2)} \qquad W^{(1)} = W^{(2)} \tag{3}$$

$$\bar{T}^{(1)} = \bar{T}^{(2)} \qquad \bar{W}^{(1)} = \bar{W}^{(2)} \tag{4}$$

After modular transformation these defects are given by operators X, satisfying relations:

$$[L_n, X] = [\bar{L}_n, X] = 0 \tag{5}$$

$$[W_n, X] = [\bar{W}_n, X] = 0 \tag{6}$$

As in the case of the boundary conditions, there are also consistency conditions, analogous to the Cardy and Cardy-Lewellen constraints, which must be satisfied by the operator X. For simplicity we shall write all the formulae for diagonal models $\overline{i} = i^*$. To formulate these conditions, one first note that as consequence of (5) and (6) X is a sum of projectors

$$X = \sum_i D^i P^i \tag{7}$$

where

$$P^i = \sum_{N,\bar{N}} (|i, N\rangle \otimes |i^*, \bar{N}\rangle)(\langle i, N| \otimes \langle i^*, \bar{N}|) \tag{8}$$

An analogue of the Cardy condition for defects requires that partition function with insertion of a pair defects after modular transformation can be expressed as sum of characters with non-negative integers. For diagonal models one can solve this condition taking for each primary a

$$D_a^i = \frac{S_{ai}}{S_{0i}} \tag{9}$$

for which one has:

$$Z_{ab} = \mathrm{Tr}\left(X_a^\dagger X_b \tilde{q}^{L_0 - \frac{c}{24}} \tilde{\bar{q}}^{\bar{L}_0 - \frac{c}{24}} \right) = \sum_{k, i\bar{i}} N_{bk}^a N_{i\bar{i}}^k \chi_i(q) \chi_{\bar{i}}(\bar{q}) \tag{10}$$

Associativity of the 4-point functions with the defect insertion implies classifying algebra condition.

For self-conjugate models $i = i^*$ without multiplicities $N_{bk}^a = 0, 1$ it reads:

$$\sum_k (C_{ij}^k)^2 C_{kk}^1 D^k \left(F_{k0} \left[\begin{array}{cc} i & i \\ j & j \end{array} \right] \right)^2 = C_{ii}^1 D^i C_{jj}^1 D^j \tag{11}$$

Liouville theory

Let us review basic facts on the Liouville field theory. Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab} \partial_a \varphi \partial_b \varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi} R\varphi \qquad (12)$$

where R is associated curvature. This theory is conformal invariant if the coupling constant b is related with the background charge Q as

$$Q = b + \frac{1}{b} \qquad (13)$$

The symmetry algebra of this conformal field theory is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m} \quad (14)$$

with central charge

$$c_L = 1 + 6Q^2 \quad (15)$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q - \alpha) \quad (16)$$

The spectrum of the Liouville theory is believed to be of the following form

$$\mathcal{H} = \int_0^\infty dp \; R_{\frac{Q}{2}+iP} \otimes R_{\frac{Q}{2}+iP} \quad (17)$$

where R_α is the highest weight representation with respect to Virasoro alegbra. Characters of the representations $R_{\frac{Q}{2}+iP}$ are

$$\chi_P(\tau) = \frac{q^{P^2}}{\eta(\tau)} \qquad (18)$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad (19)$$

Modular transformation of (18) is well-known:

$$\chi_P(-\frac{1}{\tau}) = \sqrt{2} \int \chi_{P'}(\tau) e^{4i\pi PP'} dP' \qquad (20)$$

Degenerate representations appear at $\alpha_{m,n} = \frac{1-m}{2b} + \frac{1-n}{2} b$ and have conformal dimensions

$$\Delta_{m,n} = Q^2/4 - (m/b + nb)^2/4 \qquad (21)$$

where m, n are positive integers. At general b there is only one null-vector at the level mn. Hence the degenerate character reads:

$$\chi_{m,n}(\tau) = \frac{q^{-(m/b+nb)^2} - q^{-(m/b-nb)^2}}{\eta(\tau)} \tag{22}$$

Modular transformation of (22) is

$$\chi_{m,n}(-\frac{1}{\tau}) = 2^{3/2} \int \chi_P(\tau) \sinh \frac{2\pi mP}{b} \sinh(2\pi nbP) dP \tag{23}$$

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$$D_s(P) = \frac{\cos(4P\pi s)}{2\sinh(2\pi bP)\sinh(2P\pi/b)} = \frac{S_{sP}}{S_{(1,1),P}} \tag{24}$$

and

$$\mathcal{D}_{m,n}(P) = \frac{\sinh(2\pi mPb^{-1})\sinh(2\pi nbP)}{\sinh(2\pi bP)\sinh(2P\pi/b)} = \frac{S_{(m,n),P}}{S_{(1,1),P}} \tag{25}$$

Now one can define

$$X_s = \int_P D_s(P)\mathrm{id}_{P\otimes P} dP \tag{26}$$

and

$$X_{m,n} = \int_P \mathcal{D}_{m,n}(P)\mathrm{id}_{P\otimes P} dP \tag{27}$$

where $\mathrm{id}_{P\otimes P}$ is the identity operator on the space $R_{\frac{Q}{2}+iP} \otimes R_{\frac{Q}{2}+iP}$.