Generalized Calogero-Moser and Macdonald-Ruijsenaars systems from Cherednik algebras

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1. Generalized Calogero-Moser (CM) systems

- CM systems for Coxeter groups
- Integrability through Dunkl operators
- Generalized CM systems
- Invariant ideals for rational Cherednik algebras
- Restricted Dunkl operators and integrability of the generalized CM systems

2. Generalized Macdonald-Ruijsenaars systems

- Double Affine Hecke Algebra

- Invariant ideals and generalized MR systems

Calogero-Moser (CM) systems for Coxeter groups

Coxeter root system:

- $\mathcal{R}$ - finite collection of vectors in $\mathbb{R}^N$

- $s_\alpha \mathcal{R} = \mathcal{R}\ \forall \alpha \in \mathcal{R}$,

  s_α - orthogonal reflection with respect to the hyperplane $\Pi_\alpha : (\alpha, x) = 0$

  (here $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$)

- $\mathcal{R} = \mathcal{R}_+ \bigsqcup (-\mathcal{R}_+)$, $\mathcal{R}_+$ - pairwise non-collinear

Then $\mathcal{R}$ - Coxeter root system;

$W = \langle s_\alpha | \alpha \in \mathcal{R} \rangle$ - finite Coxeter group

Ring of invariants $\mathbb{C}[x]^W = \mathbb{C}[\sigma_1, \ldots, \sigma_N]$, σ_i - basic invariants

Example $\mathcal{R} = \mathcal{A}_{N-1} \subset \mathbb{R}^N$

$\mathcal{R} = \{\pm(e_i - e_j) | 1 \leq i < j \leq N\}$

$W = S_N$

$\mathbb{C}[x]^{S_N} = \mathbb{C}[\sigma_1, \ldots, \sigma_N]$, with e.g. $\sigma_k = \sum_{i=1}^N x_i^k$

CM operator

$$L = L_c^{\mathcal{R}} = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{2c_\alpha}{(\alpha, x)} \partial_\alpha$$

where $c : \mathcal{R} \to \mathbb{C}$ is W - invariant

Example $\mathcal{R} = \mathcal{A}_{N-1}$

$W = S_N$, $c = const$,

$$L = L_c^{\mathcal{A}_{N-1}} = \Delta - \sum_{i<j}^{N} \frac{2c}{x_i - x_j} (\partial_i - \partial_j).$$

Potential gauge:

$$\widetilde{L} = f L f^{-1} = \Delta - \sum_{i<j}^{N} \frac{2c(c+1)}{(x_i - x_j)^2},$$

$f = \prod_{i<j}^{N} (x_i - x_j)^{-c}$.

Dunkl operators

$$\nabla_\xi = \nabla_\xi^{\mathcal{R},c} = \partial_\xi - \sum_{\alpha\in\mathcal{R}_+} \frac{c_\alpha(\alpha,\xi)}{(\alpha,x)}(1-s_\alpha), \quad \xi \in \mathbb{R}^N$$

Commutativity: $[\nabla_\xi, \nabla_\eta] = 0 \quad \forall \xi, \eta \in \mathbb{R}^N$ Dunkl'89

Integrability of CM systems

$$\sum_{i=1}^N \nabla_i^2 = \Delta - \sum_{\alpha\in\mathcal{R}_+} \frac{2c_\alpha}{(\alpha,x)}\partial_\alpha + \sum_{\alpha\in\mathcal{R}_+} \frac{c_\alpha(\alpha,\alpha)(1-s_\alpha)}{(\alpha,x)^2},$$

where $\nabla_i = \nabla_{e_i}$.

$$\textstyle\sum_{i=1}^N \nabla_i^2|_{\mathbb{C}[x]^W} = L \quad (s_\alpha \to 1) \text{ Heckman'91}$$

Take $g, h \in \mathbb{C}[x]^W$. Then

$$[g(\nabla)|_{\mathbb{C}[x]^W}, h(\nabla)|_{\mathbb{C}[x]^W}] = 0,$$

$\{g(x)|_{\mathbb{C}[x]^W} | g \in \mathbb{C}[x]^W\}$ – commuting family of differential operators, contains L

Rational Cherednik algebra

$\mathcal{R}$ - Coxeter root system, W - the corresponding Coxeter group, $c : \mathcal{R} \to \mathbb{C}$ - W-invariant multiplicity of the roots

Rational Cherednik algebra $H = H_c^{\mathcal{R}}$

is given by its faithful representation in $\mathbb{C}[x]$:

$H_c^{\mathcal{R}} \cong \langle \mathbb{C}[x], \mathbb{C}[\nabla], \mathbb{C}W \rangle$

$\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_N], \mathbb{C}[\nabla] = \mathbb{C}[\nabla_1, \ldots, \nabla_N]$,
$\mathbb{C}W$ - group algebra

$$p(x) : q(x) \to p(x)q(x) \quad \forall p(x) \in \mathbb{C}[x]$$

$$\nabla_\xi : q(x) \to \nabla_\xi q(x) \quad \forall \xi \in \mathbb{R}^N$$

$$w : q(x) \to q(w^{-1}x) \quad \forall w \in W$$

$\forall q \in \mathbb{C}[x]$

Generalised CM systems

$$L = \Delta - \sum_{\alpha \in A} \frac{2c_\alpha}{(\alpha, x)} \partial_\alpha,$$

$A \subset \mathbb{R}^N$ - some collection of vectors; $c_\alpha \in \mathbb{C}$ – multiplicities

L – integrable: $\exists L_1 = L, L_2, \ldots, L_N$ such that $[L_i, L_j] = 0$, $\{L_i\}$ algebraically independent

Example

$$L_k^{n,m} = \Delta - 2k \sum_{i<j}^{n} \frac{\partial_{x_i} - \partial_{x_j}}{x_i - x_j} - \frac{2}{k} \sum_{i<j}^{m} \frac{\partial_{y_i} - \partial_{y_j}}{y_i - y_j} -$$

$$2 \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \frac{\partial_{x_i} - \sqrt{k}\partial_{y_j}}{x_i - \sqrt{k} y_j}$$

Potential gauge: pairwise interaction of $n+m$ particles with inverse square potential. Mass of x particles $= 1$, mass of y particles $= 1/k$.

Origins and history

- Multidimensional Baker–Akhiezer functions and theory of Huygens' principle , $m = 1$
  Chalykh, Veselov, F'96

- Superalgebras Lie, $k = -1/2$

  Sergeev'01

- Integrability by computations

  Sergeev, Veselov'04

- Generalized discriminants Sergeev, Veselov'05

$$ {}^{Trig}_{Sym} L_k^{n,m} = {}^{Trig}_{Sym} \mathcal{L}^{\infty}|_{gener.discr.} $$

- Special solutions Hallnäs; Langmann'07

\+ **B**-family

$\boldsymbol{H}_c^{\mathcal{R}}$ - invariant ideals of $\mathbb{C}[x]$

- $P \subset W$ - parabolic subgroup up to conjugation $P = \langle s_\alpha | \alpha \in \Gamma_0 \rangle$, where $\Gamma_0 \subset \Gamma$, subgraph of the Coxeter graph for W

- D_P - intersection of mirrors, $D_P = \bigcap_{\alpha \in \Gamma_0} \Pi_\alpha$, where Π_α are hyperplanes $\{(\alpha, x) = 0\}$

- The orbit $\mathcal{D}_P = \bigcup_{w \in W} w(D_P)$

- Ideal $I_P = \{q \in \mathbb{C}[x] | q|_{\mathcal{D}_\mathcal{P}} = 0\}$

When is ideal I_P invariant under the rational Cherednik algebra $H_c^{\mathcal{R}}$?

Equivalently, $\nabla_\xi : I_P \to I_P \ \forall \xi \in \mathbb{R}^N$

Generalized Coxeter number $h^c = h^c_{\mathcal{R}}$ is defined by

$$\sum_{\alpha\in\mathcal{R}} \frac{c_\alpha(\alpha,u)(\alpha,v)}{(\alpha,\alpha)} = h^c \cdot (u,v) \quad \forall u,v \in \mathbb{R}^N$$

When $c = 1$, $h^1 = h = max_i(d_i)$ - the usual Coxeter number for $\mathcal{R}$.

Theorem F'08 *Let Coxeter graph*

$$\Gamma_0 = \Gamma_1 \sqcup \ldots \sqcup \Gamma_s,$$

where $\Gamma_1, \ldots, \Gamma_s$ *- connected. Let* $\mathcal{R}_i$ *be the Coxeter root system with the graph* Γ_i*. Then ideal* I_P *is* $H_c^{\mathcal{R}}$*-invariant if and only if*

$$h^c_{\mathcal{R}_i} = 1 \quad \forall i = 1, \ldots, s.$$

Note when $c = const$ the condition is $h = 1/c$ for all root systems $\mathcal{R}_i$, $i = 1, \ldots, s$.

Example $\mathcal{R} = \mathcal{A}_{N-1}$

$h(\mathcal{A}_{k-1}) = k$, hence

$\Gamma_0 = \Gamma_1 \sqcup \ldots \sqcup \Gamma_s$,

with $\mathcal{R}_i = A_{k-1}$ $\forall i = 1, \ldots, s$,

where $k = 1/c$.

$D_P = \{x_1 = \ldots = x_k, x_{k+1} = \ldots = x_{2k}, \ldots\}$

$\mathcal{D}_P$ - S_N-orbit of D_P.

Note that it is necessary that c is inverse integer for the existence of the invariant ideal.

More generally defining representation $\mathbb{C}[x]$ for $H_c^{\mathcal{R}}$ with $c = const$ is reducible if and only if

$$c = \frac{n}{d_i} + m$$

for some integer m, n, $1 \leq n \leq d_i - 1$, $m \geq 0$, and some degree d_i. Dunkl, de Jeu, Opdam'94

Restricted Dunkl operators and integrable systems of CM type

Let $I_P = \{q \in \mathbb{C}[x] | q|_{\mathcal{D}_\mathcal{P}} = 0\}$ be $H_c^{\mathcal{R}}$-invariant

Then $\nabla_\xi|_{\mathcal{D}_\mathcal{P}}$ is correctly defined.

Theorem F'08 *The operator*

$$\sum_{i=1}^{N} \nabla_i^2|_{\mathbb{C}[x]^W|_{\mathcal{D}_\mathcal{P}}}$$

is integrable;

$$[g(\nabla)|_{\mathbb{C}[x]^W|_{\mathcal{D}_\mathcal{P}}}, f(\nabla)|_{\mathbb{C}[x]^W|_{\mathcal{D}_\mathcal{P}}}] = 0,$$

$\forall g, f \in \mathbb{C}[x]^W$.

Explicit form:

$$\sum_{i=1}^{N} \nabla_i^2|_{\mathbb{C}[x]^W|_{\mathcal{D}_\mathcal{P}}} = \Delta_z - \sum_{\substack{\alpha \in \mathcal{R}_+ \\ \widehat{\alpha} \neq 0}} \frac{2c_\alpha}{(\widehat{\alpha}, z)} \partial_{\widehat{\alpha}},$$

where Δ_z *is Laplacian on* D_P, *and* $\widehat{\alpha}$ *is projection of* α *onto* D_P (z *are coordinates on* D_P).

This construction gives Sergeev-Veselov operators (at special parameters) starting from A, B, D root systems. Exceptional groups lead to new integrable operators.

Example $\mathcal{R} = F_4 =$

$= \{\pm e_i, 1 \le i \le 4; \pm e_i \pm e_j, 1 \le i < j \le 4;\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$

$c(e_i) = c(\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)) = \frac{1}{2},\ c(e_i \pm e_j) = c$

$\Gamma_0 = \bullet \cong \mathcal{A}_1\ (P \cong \mathbb{Z}_2)$

$$L = \partial_1^2 + \partial_2^2 + \partial_3^2 - \sum_{i=1}^{3} \frac{(4c+1)\partial_i}{x_i} - \sum_{\substack{i,j=1 \\ i<j}}^{3} \frac{2c(\partial_i \pm \partial_j)}{x_i \pm x_j} -$$

$$- \sum_{\pm} \frac{2(\partial_1 \pm \partial_2 \pm \partial_3)}{x_1 \pm x_2 \pm x_3}.$$

Double Affine Hecke Algebra Cherednik'92

$\mathcal{R}$ - crystallographic root system of rank N (with minuscule coweight)

$$\mathcal{H}_{q,t}^{\mathcal{R}} = \langle X^{\lambda}, T_i, \pi_r \rangle / relations$$

- $\lambda \in \mathcal{P}$ - weight lattice: $(\alpha^{\vee}, \lambda) \in \mathbb{Z} \quad \forall \alpha \in \mathcal{R}$

  here $\alpha^{\vee} = \frac{2\alpha}{(\alpha,\alpha)}$

- $i = 1, \ldots, N$, $\langle T_i \rangle$ - Hecke algebra

- $r \subset \{1, \ldots, N\}$: $\pi_r \leftrightarrow b_r$ - minuscule coweight

  $b \in \mathcal{P}^{\vee}$ - minuscule: $(\alpha, b) \in \{0, 1\} \quad \forall \alpha \in \mathcal{R}_{+}$

Faithful representation of $\mathcal{H}_{q,t}^{\mathcal{R}}$

The space of representation is $\mathbb{C}[e^{(\mu,x)}], \mu \in \mathcal{P}$

Let $\Delta = \{\alpha_1, \dots, \alpha_N\}$ be simple roots; let $t : \mathcal{R} \to \mathbb{C}$ be W-invariant, denote $t_i = t_{\alpha_i} = t(\alpha_i)$.

Define the representation:

$$X^\lambda \to e^{(\lambda,x)}, \quad T_i \to t_i + \frac{t_i^{-1} - t_i e^{-(\alpha_i,x)}}{1 - e^{-(\alpha_i,x)}}(s_{\alpha_i} - 1),$$

$$\pi_r \to \tau(\hbar b_r) u_r$$

$\tau(\cdot)$-shift operator: $\tau(v)f(x) = f(x-v)$; $q = e^{\hbar/2}$

$u_r \in W$ is such that

$$u_r : \Delta \cup \{-\theta\} \to \Delta \cup \{-\theta\},$$

and $u_r(-\theta) = \alpha_r$ (θ - maximal root).

Cherednik-Dunkl operators

Let $b_1, \ldots, b_N$ be fundamental coweights: $(b_i, \alpha_j) = \delta_{ij}$. Consider reduced decomposition

$$\tau(\hbar b_i) = \pi_r s_{i_1} \cdots s_{i_l}$$

in the extended Weyl group $\widehat{W} = W \ltimes \tau(\hbar \mathcal{P}^\vee)$

(reduced: minimal length, $l(\widehat{w}) = |\widehat{R}_- \cap \widehat{w}\widehat{R}_+|$, where $\widehat{R}_+ = -\widehat{R}_- = R_+ \cup \{\alpha + \hbar k \mid \alpha \in R, k \in \mathbb{Z}_{>0}\}$)

Then define $T_{\tau(\hbar b_i)} = \pi_r T_{i_1} \ldots T_{i_l}$. One has $[T_{\tau(\hbar b_i)}, T_{\tau(\hbar b_j)}] = 0$.

Cherednik-Dunkl operators

For any $\lambda \in \mathcal{P}^\vee$, $\quad \lambda = \sum_{i=1}^{N} m_i b_i, \quad m_i \in \mathbb{Z}$, define

$$Y^\lambda = \prod_{i=1}^{N} (T_{\tau(b_i)})^{m_i}.$$

Then commutativity holds: $Y^\lambda Y^\mu = Y^\mu Y^\lambda = Y^{\lambda+\mu} \quad \forall \lambda, \mu \in \mathcal{P}^\vee$.

Macdonald-Ruijsenaars (MR) operators

For any $\lambda \in \mathcal{P}^\vee$ define the difference operator $M_\lambda = \sum_{\nu \in W(-\lambda)} Y^\nu|_{W-invariants}$

Then $[M_\lambda, M_\mu] = 0 \quad \forall \lambda, \mu \in \mathcal{P}^\vee$.

Let b_r be a minuscule coweight. The corresponding MR operator is

$$M_{b_r} = \sum_{\lambda \in W(-b_r)} \Bigg(\prod_{\substack{\alpha \in R \\ (\alpha,\lambda)=1}} \frac{t_\alpha e^{(\alpha,x)} - t_\alpha^{-1}}{e^{(\alpha,x)} - 1} \Bigg) \tau(\hbar\lambda).$$

For the coweight $\theta^\vee$ one has

$$M_{\vartheta^\vee} = \sum_{\beta \in W\vartheta} A_\beta \Big(\tau(\hbar\beta^\vee) - 1 \Big)$$

where

$$A_\beta = \Bigg(\prod_{\substack{\alpha \in R \\ (\alpha,\beta^\vee)>0}} \frac{t_\alpha e^{(\alpha,x)} - t_\alpha^{-1}}{e^{(\alpha,x)} - 1} \Bigg) \frac{q^{-1} t_0 e^{(\beta,x)} - q t_0^{-1}}{q^{-1} e^{(\beta,x)} - q}$$

$\mathcal{H}_{q,t}^{\mathcal{R}}$ - invariant ideals of $\mathbb{C}[e^{(\mu,x)}]$, $\mu \in \mathcal{P}$

$\mathcal{A}_n, \mathcal{C}_n$ cases studied by Kasatani'05,'08

For general $\mathcal{R}$ we define

- $P = \langle s_\alpha | \alpha \in \Gamma_0 \rangle \subset W$ - parabolic subgroup; $\Gamma_0 \subset \Gamma$ - subgraph of the Coxeter graph of W; $\Gamma_0 = \Gamma_1 \sqcup \ldots \sqcup \Gamma_s$; $\Gamma_1, \ldots, \Gamma_s$ - connected components, $\mathcal{R}_i$ - corresponding root systems

- $D_P := \bigcap_{\alpha \in \Gamma_0} \Pi_\alpha^\eta$, where Π_α^η are hyperplanes $\{(\alpha, x) = \eta\}$

- The union of planes $\mathcal{D}_\mathcal{P} = \bigcup_{w \in W_+} w(D_P)$, where

  $W_+ := \{w \in W | w(\alpha) \in \mathcal{R}_+ \quad \forall \alpha \in \Gamma_0\}$

- The ideal $I_P := \{q \in \mathbb{C}[e^{(\mu,x)}] |\ q|_{\mathcal{D}_\mathcal{P}} = 0\}$

Theorem F., Silantyev'09 Ideal I_P is invariant under the DAHA $\mathcal{H}_{q,t}^{\mathcal{R}}$ if

$$h^\eta(\mathcal{R}_i) = \hbar \quad \forall i = 1, \ldots, s,$$

and $t = e^{\eta/2}$.

Note in the case $\eta = const$ the condition on η is $h(\mathcal{R}_i) = \frac{\hbar}{\eta} \quad \forall i = 1, \ldots, s.$

This leads to integrable generalized Macdonald-Ruijsenaars (MR) operators - restrictions of usual MR operators: $[M_\lambda|_{D_P}, M_\mu|D_P] = 0$. Explicit operator for the minuscule coweight is

$$M_{b_r}|_{D_P} = \sum_{w \in W} \prod_{\substack{\alpha \in \mathcal{R} \\ (\alpha, wb_r) = -1}} \frac{te^{(\alpha,x)} - t^{-1}}{e^{(\alpha,x)} - 1}|_{D_P} \tau(-\hbar \widehat{wb_r}),$$

where $\hat{u}$ is projection of u on D_P, $x \in D_P$.

These operators are known for classical root systems and new for exceptional root systems.

Example $\mathcal{R} = \mathcal{A}_{N-1}$, rational version

The MR operator corresponding to $b_r = e_1$ is

$$M = \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{x_i - x_j + \hbar}{x_i - x_j} T_{x_i}^{\hbar},$$

where $T_{x_i}^{\hbar} = \tau(\hbar e_i)$. We take $\Gamma_0 = \mathcal{A}_{k-1} \sqcup \ldots \sqcup \mathcal{A}_{k-1}$.

The restricted operator is

$$M^{n,m} = \hbar \sum_{i=1}^{m} \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{y_i - y_j + \hbar}{y_i - y_j} \prod_{j=1}^{n} \frac{y_i - x_j + \eta}{y_i - x_j} T_{y_i}^{\eta} +$$

$$\eta \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \prod_{j=1}^{m} \frac{x_i - y_j + \hbar}{x_i - y_j} T_{x_i}^{\hbar}$$

with $\hbar = k\eta, N = mk + n$.

Sergeev, Veselov'08 Integrability for arbitrary $\hbar, \eta$; $M_{trig}^{n,m}$ as a restriction from $\mathcal{M}^{\infty}$.

DAHA of $C^\vee C$ type and generalized Macdonald-Koornwinder operators

Let $R = C_N =$

$$\{2e_i, 1 \le i \le N;\ \ \pm e_i \pm e_j, 1 \le i < j \le N\},$$

$\alpha_1, \ldots, \alpha_N$ - simple roots. Faithful representation of DAHA by operators acting in $\mathbb{C}[e^{(\mu,x)}]$, $\mu \in \mathcal{P}$

$$X^\mu = e^{(\mu,x)},\ \mu \in \mathcal{P}$$

$$T_i = \frac{(1 - t_i u_i e^{-(\alpha_i,x)/2})(1 + t_i u_i^{-1} e^{-(\alpha_i,x)/2})}{t_i(1 - e^{-(\alpha_i,x)})}(s_{\alpha_i} - 1)$$

$$+ t_i, \qquad i = 0, 1, \ldots, N.$$

α_0 - affine root, $(\alpha_0, x) = -(\theta, x) + \hbar$.

t, u - W-invariant multiplicities on $\widehat{C}_N$ - affine root system; $t_i = t(\alpha_i)$, $u_i = u(\alpha_i)$, with $u(e_i \pm e_j) = 1$.

Overall 5 parameters: t_0, u_0, $t_1 = t(e_i \pm e_j)$, $t_n = t(\pm 2e_i)$, $u_n = u(\pm 2e_i)$.

This leads to Macdonald-Ruijsenaars operator Noumi'94

$$M_{\vartheta^\vee} = \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^{n} A_{i,\epsilon}\Big(\mathcal{T}_{x_i}^{-\epsilon\hbar} - 1\Big).$$

where $\mathcal{T}_{x_i}^{-\epsilon\hbar} = \tau(\epsilon\hbar e_i)$ is the shift operator, and

$$A_{i,\epsilon} = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(t_1 e^{\epsilon x_i - x_j} - t_1^{-1})(t_1 e^{\epsilon x_i + x_j} - t_1^{-1})}{(e^{\epsilon x_i + x_j} - 1)(e^{\epsilon x_i - x_j} - 1)} \times$$

$$\frac{(1 - t_0 u_0 q^{-1} e^{\epsilon x_i})(1 + t_0 u_0^{-1} q^{-1} e^{\epsilon x_i})}{(1 - q^{-2} e^{2\epsilon x_i})} \times$$

$$\frac{(1 - t_n u_n e^{\epsilon x_i})(1 + t_n u_n^{-1} e^{\epsilon x_i})}{(1 - e^{2\epsilon x_i})},$$

where $g(z) = \frac{t_1 e^z - t_1^{-1}}{e^z - 1}$, $q = e^{\hbar/2}$.

For appropriate invariant ideal I_P we get the generalized MR operator F., Silantyev'10

$$M_{\theta^\vee}|_{D_P} = \sum_{\substack{i=1\\ \epsilon=\pm 1}}^{N_1} A_{i,\epsilon}\Big(\mathcal{T}_{x_i}^{-\epsilon\hbar}-1\Big)+\sum_{\substack{\ell=1\\ \epsilon=\pm 1}}^{N_2} B_{\ell,\epsilon}\Big(\mathcal{T}_{y_\ell}^{-\epsilon\eta}-1\Big),$$

where

$$A_{i,\epsilon} = (t_0t_n)^{-1}\prod_{\substack{j=1\\ j\neq i}}^{N_1} g(\epsilon x_i - x_j; t_1, 1)g(\epsilon x_i + x_j; t_1, 1)\times$$

$$\prod_{\ell=1}^{N_2} g(\epsilon x_i - y_\ell; (t_1 q)^{1/2}, (t_1/q)^{1/2})\times$$

$$\prod_{\ell=1}^{N_2} g(\epsilon x_i + y_\ell; (t_1 q)^{1/2}, (t_1/q)^{1/2})\times$$

$$\frac{(1-t_0u_0q^{-1}e^{\epsilon x_i})(1+t_0u_0^{-1}q^{-1}e^{\epsilon x_i})(1-t_nu_ne^{\epsilon x_i})}{(1-q^{-2}e^{2\epsilon x_i})(1-e^{2\epsilon x_i})(1+t_nu_n^{-1}e^{\epsilon x_i})^{-1}}$$

where $g(z;a,b) = \frac{ae^z-a^{-1}}{be^z-b^{-1}}$, $q = e^{\hbar/2} = t_1^k$, $\hbar = k\eta$, $k \in Z_+$.

$$B_{\ell,\epsilon} = \frac{q - q^{-1}}{(t_1 - t_1^{-1})(t_0 t_n)} \prod_{j=1}^{N_1} g(\epsilon y_\ell - x_j; (qt_1)^{1/2}, (q/t_1)^{1/
$$

$$\prod_{j=1}^{N_1} g(\epsilon y_\ell + x_j; (qt_1)^{1/2}, (q/t_1)^{1/2}) \times$$

$$\prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{N_2} \Big(g(\epsilon y_\ell - y_{\ell'}; q, 1) g(\epsilon y_\ell + y_{\ell'}; q, 1) \Big) \times$$

$$\frac{(1 - t_0 u_0 t_1^{-1} e^{\epsilon y_\ell})(1 + t_0 u_0^{-1} t_1^{-1} e^{\epsilon y_\ell})}{(1 - t_1^{-2} e^{2\epsilon y_\ell})} \times$$

$$\frac{(1 - t_n u_n q t_1^{-1} e^{\epsilon y_\ell})(1 + t_n u_n^{-1} q t_1^{-1} e^{\epsilon y_\ell})}{(q t_1^{-1} - q t_1^{-1} e^{2\epsilon y_\ell})}$$

Thank you for your attention!