hep-th/yymmnnn

Noncommutative Gravity with Torsion and Vortex Solitons

Mohab Abou-Zeid[1] and Olaf Lechtenfeld[2]

Institut für Theoretische Physik, Leibniz Universität Hannover,
Appelstrasse 2, 30167 Hannover, Germany

Abstract

A Lorentzian theory of noncommutative gravity with torsion coupled to $U(1)_\star$ Chern-Simons gauge fields and charged matter is investigated. The construction is a generalisation of the well-known relationship between three-dimensional gravity and Chern-Simons theory and it can be viewed as a toy model for a noncommutative deformation of the $U(2) \times U(2)$ Aharony-Bergman-Jafferis-Maldacena model. Utilising the Harvey-Kraus-Larsen solution-generating technique, BPS solutions of the field equations are constructed which are exact for all values of the noncommutativity parameter θ, regular for $\theta \neq 0$ and singular in the limit $\theta \to 0$. The gravitational interpretation of the solutions is discussed.

[1]mohab.abouzeid@itp.uni-hannover.de
[2]lechtenf@itp.uni-hannover.de

1 Noncommutative Field Theory

We start by recalling some facts about noncommutative field theories and setting up the notation. We then briefly review Chern-Simons theory and its relation to gravity with torsion.

Consider a flat noncommutative space with coordinates (t, x, y). Time t is taken as a real (commuting) parameter while the operators associated to x and y satisfy

$$[\hat{x}, \hat{y}] = -i\theta, \tag{1.1}$$

where θ is a commuting parameter of length dimension 2. There is a one-to-one Moyal-Weyl correspondence between operators on Hilbert space and ordinary functions: for a given function f with Fourier decomposition

$$f(x, y) = \int \frac{d^2k}{(2\pi)^2} \tilde{f}(k) e^{i(k_x x + k_y y)}, \tag{1.2}$$

the corresponding operator $\hat{f}$ is the Weyl ordering of the formal expression

$$\hat{f}(\hat{x}, \hat{y}) = \int \frac{d^2k}{(2\pi)^2} \tilde{f}(k) e^{i(k_x \hat{x} + k_y \hat{y})}. \tag{1.3}$$

The operator product $\hat{f} \cdot \hat{g}$ of two operators $\hat{f}, \hat{g}$ corresponds to the $\star$ product of functions f, g defined by

$$f \star g(x, y) \equiv e^{-\frac{i}{2}\theta\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2}\right)} f(x_1, y_1) g(x_2, y_2)|_{x_1 = x_2, y_1 = y_2} \tag{1.4}$$

Noncommutative field theory Lagrangians can be written in the operator framework or in terms of commutative fields for which the ordinary product is replaced by the $\star$ product defined in (1.4); we will be using both approaches.

A basis for the Hilbert space is given by the eigenstates $\{|n\rangle\}$ of the creation and annihilation operators defined by

$$\hat{a} = \frac{\hat{x} - i\hat{y}}{\sqrt{2\theta}}, \qquad \hat{a}^\dagger = \frac{\hat{x} + i\hat{y}}{\sqrt{2\theta}} \tag{1.5}$$

in such a way that they satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The projection operator $|n\rangle\langle n|$ is mapped to the function $2(-1)^n L_n(\frac{2r^2}{\theta}) e^{-\frac{r^2}{\theta}}$, where $L_n(z)$ is the n-th order Laguerre polynomial. Any operator $\hat{O}$ can be expressed as a linear combination $\Sigma_{mn} O_{mn} |m\rangle\langle n|$ for some complex coefficients O_{mn}. For example, the number operator $\hat{N} = \hat{a}\hat{a}^\dagger$ is diagonalised by the basis states $\{|n\rangle\}$ and since its eigenvalues $N \sim \frac{1}{2\theta}(x^2 + y^2)$, it measures the square of the distance to the origin.

Under the Moyal-Weyl mapping, integration over space becomes the operator trace, which we denote by Tr:

$$\int d^2xf \longleftrightarrow 2\pi\theta \mathrm{Tr}\hat{f};$$

derivatives are related to commutators as follows:

$$\partial_x f \to -\frac{i}{\theta}[\hat{y},\hat{f}], \qquad \partial_y f \to \frac{i}{\theta}[\hat{x},\hat{f}],$$

i. e.

$$\partial_z f \to -\frac{1}{\sqrt{\theta}}[\hat{a}^\dagger,\cdot], \qquad \partial_{\bar{z}} f \to \frac{1}{\sqrt{\theta}}[\hat{a},\cdot]$$

in the complex coordinates defined by $z = \frac{1}{\sqrt{2}}(x+iy) = \sqrt{\theta}\hat{a}$. In the following, for the sake of simplicity, we often drop the hats on operators.

2 Noncommutative Chern-Simons and Gravity with Torsion

Consider noncommutative Chern-Simons theory with gauge group G and action

$$S_{CS} = \int dt 2\pi\theta \mathrm{Tr}\left[\frac{\kappa}{2}\epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right)\right], \tag{2.1}$$

where tr denotes the trace over group indices and κ is a coupling related to the Chern-Simons level k by $\kappa = \frac{k}{2\pi}$. As explained in [24, 25], gauge invariance requires κ to be quantised if $\Pi_1(G)$ is nontrivial, as is the case for $U(n)$ (remarkably, this includes the case of the noncommutative $U(1)$ theory). The action is then invariant under the noncommutative gauge transformations

$$A_\mu \to U^\dagger A_\mu U + iU^\dagger\partial_\mu U, \tag{2.2}$$

where U is a t-dependent element of G which acts as the identity on the $\{|n\rangle\}$ as $n\to\infty$. Following [28, 20], the construction of ref. [12] involves considering the difference

$$S_{LR} = S^L_{CS} - S^R_{CS} \tag{2.3}$$

of the actions for two noncommutative Chern-Simons gauge gauge fields A^L and A^R with the same coupling κ and a product gauge group $G = U(1,1)_\star\times U(1,1)_\star$. Note that a single Chern-Simons term (2.1) breaks parity, whereas the difference (2.3) is parity invariant; parity invariance is unbroken in gravity, so at least two such terms are needed in the construction. The choice of gauge group G is natural for a noncommutative generalisation of the relationship between three dimensional gravity and $SO(2,1) \times SO(2,1)$ Chern-Simons theory in a Lorentzian space [28, 20][3]. An element M of $U(1,1)_\star$ satisfies

$$M^{-1} = \eta M^\dagger \eta, \qquad M \star M^{-1} = M^{-1} \star M = 1, \tag{2.4}$$

where $\eta = \mathrm{diag}(-1,1)$. All fields can be chosen to be real and their reality is preserved under noncommutative gauge transformations. Write

$$\begin{aligned} A &= \omega + \frac{1}{l}e + \frac{i}{2}b \\ \tilde{A} &= \omega - \frac{1}{l}e + \frac{i}{2}\tilde{b}, \end{aligned} \tag{2.5}$$

with l a real constant of length dimension -1 and

$$\omega = \omega\tau^a, \qquad e = e^a\tau_a, \qquad b = b\mathbb{I}, \qquad \tilde{b} = \tilde{b}\mathbb{I}. \tag{2.6}$$

Here $a = 0,1,2$ and τ_A $(A = 0,1,2,3)$ are the generators of the Lie algebra $u(1,1)$. In terms of the standard Pauli matrices, the τ_A can be chosen as follows:

$$\tau_0 = \frac{i}{2}\sigma_3, \qquad \tau_1 = \sigma_1, \qquad \tau_2 = \frac{1}{2}\sigma_2, \qquad \tau_3 = \frac{i}{2}\mathbb{I} \tag{2.7}$$

and the normalisation is

$$\mathrm{tr}\,(\tau_A\tau_B) = \frac{1}{2}\eta_{AB}, \tag{2.8}$$

where $\eta_{AB} = \mathrm{diag}(-1,1,1,-1)$ is the $u(1,1)$ inner product. The action (2.3) can then be written as

$$\begin{aligned} S_{ECS} = &\frac{1}{l}\int \epsilon_{abc}\left(e^a \wedge R^{bc} - \frac{1}{3l^2}e^a \wedge e^b \wedge e^c\right) \\ &-\frac{1}{2}\int (b \wedge db - b \wedge b \wedge b) + \frac{1}{2}\int \left(\tilde{b} \wedge d\tilde{b} - \tilde{b} \wedge \tilde{b} \wedge \tilde{b}\right) \\ &+\frac{i}{2l}\int \eta_{ab}\left(e^a \wedge \omega^b + \omega^a \wedge e^b\right) \wedge \left(b + \tilde{b}\right) \\ &+\frac{i}{2l}\int \eta_{ab}\left(\omega^a \wedge \omega^b + \frac{1}{l^2}e^a \wedge e^b\right) \wedge \left(b - \tilde{b}\right) \end{aligned} \tag{2.9}$$

(we use units in which $16\pi G_N = 1$, where G_N denotes the Newton constant). Here all wedge products are taken with respect to the $\star$ product and

$$\begin{aligned} \mathcal{R} &= d\omega + \omega \wedge \omega \\ \mathcal{T} &= de + \omega \wedge e + e \wedge \omega. \end{aligned} \tag{2.10}$$

[3]A closely related Euclidean theory of noncommutative gravity based on the complex group $GL(2,\mathbb{C})$ was discussed in ref. [7].

If ω is interpreted as the Levi-Civita connection and the $U(1,1)_\star$ valued fields $\mathcal{R}$ and $\mathcal{T}$ are expanded as

$$\begin{aligned}\mathcal{R} &= R^a\tau_a - \frac{i}{2}\eta_{ab}\omega^a \wedge \omega^b\tau_3 \\ \mathcal{T} &= T^a\tau_a - \frac{i}{2}\left(\omega^a \wedge e_a + e^a \wedge \omega_a\right)\tau_3, \end{aligned} \tag{2.11}$$

then the expressions (2.10) yield formulae for the quantities R^a and T^a which are non-commutative generalisations of the familiar expressions for the curvature and torsion, viz.

$$\begin{aligned}R^a &= d\omega^a - \frac{1}{2}\epsilon^a{}_{bc}\omega^a \wedge \omega^b \\ T^a &= de^a + \frac{1}{2}\left(\omega^a{}_b \wedge e^b + \omega^b \wedge \omega_b{}^a\right), \end{aligned} \tag{2.12}$$

wherein

$$\omega^{ab} = \epsilon^{abc}\omega_c, \qquad R^{ab} = \epsilon^{abc}R_c. \tag{2.13}$$

Thus (2.3) (or (2.9)) describes a three -dimensional theory of torsionful gravity in non-commutative space with a negative cosmological constant $\Lambda = -1/l^2$ coupled to two noncommtutative $U(1)_\star$ Chern-Simons gauge fields b and $\tilde{b}$ [12]. Moreover the invariance of (2.3) under the gauge transformations (2.2) translates into transformations of the fields $\omega, e, b, \tilde{b}$ which leave the action (2.9) invariant and can be interpreted as deformed rotations, translations and $U(1) \times U(1)$ gauge transformations.

3 Adding Higgs Scalars

Let us now consider adding matter to the noncommutative Chern-Simons action (2.3). In order to study vortices and solitons in our model, we mostly need to consider charged noncommutative Higgs fields with a particular sixth order potential. The matter action will be of the form

$$S_{matter} = S_{Higgs} + S_{ferm}, \tag{3.1}$$

where the action for the noncommutative Higgs fields is

$$S_{Higgs} = \int dt 2\pi\theta \mathrm{Trtr}\left[D_\mu\phi D^\mu\phi^\dagger + V(\phi\phi^\dagger)\right] \tag{3.2}$$

and S_{ferm} denotes fermionic and other matter fields which we set to zero for the sake of simplicity. As the gauge group G is a product of two $U(1,1)_\star$ factors, we need to distinguish gauge transformations with respect to the left $U(1,1)_\star$ from gauge transformations with respect to the right $U(1,1)_\star$ and mind the noncommutativity of operator orderings. The Higgs fields can be chosen to transform in the adjoint, the fundamental or the bifundamental representations of G; here we will focus on the fundamental and bifundamental representations.

For a charged noncommutative Higgs field in the fundamental representation, we can consider the 'left module' [17, 14, 21] in which the fields transform as

$$\phi \to U^L \phi, \qquad A^L_\mu \to (U^L)^\dagger A^L_\mu U^L + i(U^L)^\dagger \partial_\mu U^L \tag{3.3}$$

and the covariant derivative is defined by

$$D_\mu \phi = \partial_\mu \phi - i A^L_\mu \phi. \tag{3.4}$$

We can also consider the 'right module' in which the fields transform as

$$\phi \to \phi U^R, \qquad A^R_\mu \to (U^R)^\dagger A^L_\mu U^R - i(U^R)^\dagger \partial_\mu U^R \tag{3.5}$$

and the covariant derivative is

$$D_\mu \phi = \partial_\mu \phi - i \phi A^R_\mu. \tag{3.6}$$

For ϕ in the bifundamental representation,

$$\phi \to U^L \phi U^R \tag{3.7}$$

and

$$D_\mu \phi = \partial_\mu \phi - i A^L_\mu \phi + i \phi A^R_\mu. \tag{3.8}$$

A special case of the bifundamental transformations is obtained by setting

$$A^L_\mu = A^R_\mu = A_\mu, \qquad D_\mu \phi = \partial_\mu \phi - i A_\mu \phi + i \phi A_\mu; \tag{3.9}$$

this corresponds to the diagonal $U(1,1)_\star$. Note that the noncommutative gauge transformations considered above reduce to the their commutative counterparts $U(1,1)_L$, $U(1,1)_R$ and $U(1,1)_{diag}$ in the limit $\theta \to 0$.

The Higgs potential is taken to be of the form

$$V(\xi) = -\frac{1}{\kappa^2} \xi \left(v^2 - \xi\right)^2. \tag{3.10}$$

This is the Bogomol'nyi limit of the general gauge invariant and renormalisable potential possessing a symmetry breaking minimum at $v = |\phi|$; in this limit the symmetric minimum is degenerate with the asymmetric one [29]. The corresponding couplings to the gravity, torsion and abelian Chern-Simons fields can be obtained by expanding the Higgs field as

$$\phi = \phi^a \tau_a + \varphi \mathbb{I} \tag{3.11}$$

and expressing the kinetic term in (3.2) in terms of ω, e, b and $\tilde{b}$ using (2.5) and (2.7)-(2.8). For a Higgs field in the bifundamental representation, this yields

$$\begin{aligned} S_{ECSH} &= \int \big[\partial_\mu\phi^a\partial^\mu\phi^\dagger_a + \partial_\mu\phi^a\varphi^\dagger\tau_a + \varphi\partial_\mu\phi^{a\dagger}\tau^\dagger_a + \varphi\varphi^\dagger\mathbb{I} + \\ &+i\left(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^\dagger_a + \partial_\mu\phi^a\varphi^\dagger\tau_a + \partial_\mu\varphi\phi^{a\dagger}\tau^\dagger_a + \partial_\mu\varphi\varphi^\dagger\mathbb{I}\right)\left(\omega + \frac{1}{l}e + \frac{i}{2}b\mathbb{I}\right) + \dots \\ &+ \dots + V(\phi\phi^\dagger)\big] \,. \end{aligned}$$

The expression (3.12) simplifies considerably for Higgs fields in the left or right module, as can be seen by setting $A^R = 0$ and $A^L = 0$, and for fields in the diagonal $U(1,1)_\star$ (for which $A^R = A^L = A$).

The action of the model which we will be investigating further below can be written alternatively as

$$S = S_{LR} + S_{Higgs}, \tag{3.12}$$

where S_{LR} is given in (2.3) and S_{Higgs} in (3.2), or more explicitly as

$$S = S_{ECS} + S_{ECSH} \tag{3.13}$$

where S_{ECS} is the CKMZ action (2.9) and S_{ECSH} is the action for the noncommutative Higgs system coupled to the noncommutative gravity, torsion and $U(1)_\star$ Chern-Simons gauge fields given in eq. (3.12).

It is interesting to compare the $\theta \to 0$ limit of this model to the bosonic part of ABJM theory [30], which was proposed as a description of multiple M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. More precisely, we need to compare with the bosonic part of the maximally supersymmetric mass deformation of the ABJM model in ref. [34]. Recall that the ABJM theory with gauge group $U(N) \times U(N)$ is an ordinary Chern-Simons-matter system with $\mathcal{N} = 6$ superconformal symmetry describing the low-energy limit of N M-branes near the singularity. In the 't Hooft limit of large N with fixed ratio N/k, this theory is conjectured to be dual to type IIA superstring theory on $AdS_4 \times \mathbb{CP}^3$. The case of particular interest to us is that of $N = 2$ and gauge group $U(2) \times U(2)$, as this is most closely related to the model studied here; for $k = 2$, this describes two M2-branes on an $\mathbb{R}^8/\mathbb{Z}_2$ orbifold [30]. The ABJM action for this case is

$$\mathcal{S}_{ABJM} = \mathcal{S}^L_{CS} + \mathcal{S}^R_{CS} + \mathcal{S}_{matter} \tag{3.14}$$

where $\mathcal{S}^L_{CS}$ and $\mathcal{S}^R_{CS}$ are the actions for two ordinary Chern-Simons gauge fields $\mathcal{A}^L, \mathcal{A}^R$ and the Chern-Simons levels associated to the two $U(2)$ factors of the gauge group are equal and opposite:

$$\mathcal{S}^L_{CS} + \mathcal{S}^R_{CS} = \frac{k}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\left[\mathrm{tr}\left(\mathcal{A}^L_\mu\partial_\nu\mathcal{A}^L_\rho - \frac{2i}{3}\mathcal{A}^L_\mu\mathcal{A}^L_\nu\mathcal{A}^L_\rho\right) - \mathrm{tr}\left(\mathcal{A}^R_\mu\partial_\nu\mathcal{A}^R_\rho - \frac{2i}{3}\mathcal{A}^R_\mu\mathcal{A}^R_\nu\mathcal{A}^R_\rho\right)\right]$$

$$(3.15)$$

and $\mathcal{S}_{matter}$ is an action for complex fields C_I and $\bar{C}^I$ as well as fermions ψ_I and $\bar{\psi}^I$ transforming in the $(\mathbf{2},\bar{\mathbf{2}})$ and $(\bar{\mathbf{2}},\mathbf{2})$ of $U(2)$ respectively; here $I=1,2,3,4$ is the $SU(4)$ R-symmetry index. This action takes the form

$$\mathcal{S}_{matter} = \int d^3x \left[\mathrm{tr}\left(D_\mu C_I D^\mu \bar{C}^I\right) + i\mathrm{tr}\left(\bar{\psi}^I\gamma^\mu D_\mu \psi_I\right)\right] + \mathcal{S}_{int}, \tag{3.16}$$

where $\gamma^\mu = (i\sigma_2, \sigma_1, \sigma_3)$ are the Dirac matrices. Here the covariant derivatives are defined by

$$D_\mu C_I = \partial_\mu C_I + i\left(\mathcal{A}^L_\mu C_I - C_I \mathcal{A}^R_\mu\right), \qquad D_\mu \bar{C}_I = \partial_\mu \bar{C}_I - i\left(\bar{C}^I\mathcal{A}^L_\mu - \mathcal{A}^R_\mu \bar{C}_I\right) \tag{3.17}$$

(with similar definitions for the fermions) and the interaction term is of the form

$$\begin{aligned} \mathcal{S}_{int} &= \frac{4\pi^2}{3k^2}\int d^3x \mathrm{tr}\left(C_I\bar{C}^I C_J \bar{C}^J C_K \bar{C}^K + C_I\bar{C}^J C_J\bar{C}^K C_K \bar{C}^I\right. \\ &\qquad \left. +4C_I\bar{C}^J C_K\bar{C}^I C_J\bar{C}^K - 6C_I\bar{C}^J C_J\bar{C}^I C_K\bar{C}^K\right) + \mathcal{S}_{Yuk} \end{aligned} \tag{3.18}$$

with $\mathcal{S}_{Yuk}$ containing $\psi^2 C^2$ Yukawa coupling terms which will not concern us further. The maximally supersymmetric deformation of ref. [34] breaks the $SU4)_R \times U(1)$ global symmetry of (3.19) to a $SU(2)\times SU(2)\times U(1)\times U(1)\times \mathbb{Z}_2$ symmetry which is preserved by the deformation

$$\mathcal{S}_{mass} = -\frac{4\pi\mu}{k}\int d^3x \mathrm{tr}\left(C_I\bar{C}^I C_J \eta^J_K \bar{C}^K - \bar{C}^I C_I \bar{C}^J \eta^J_K C_K + \frac{\mu k}{4\pi}\bar{C}^I C_I\right) - \mu \mathrm{tr}\bar{\psi}^I \eta^J_I \psi_J,$$

where μ denotes the mass deformation parameter and $\eta = \mathrm{diag}(1,1,-1,-1)$. Setting $C_1 = \phi$, $C_I = 0$ for $I = 2,3,4$ and comparing the action

$$\mathcal{S}_\mu = \mathcal{S}_{ABJM} + \mathcal{S}_{mass} \tag{3.19}$$

with the $\theta \to 0$ limit of our noncommutative Chern-Simons-Higgs system (3.12), we find that the bosonic part of the former essentially agrees with the latter, with the identification

$$v = \left(\frac{\mu k}{4\pi}\right)^{\frac{1}{4}}. \tag{3.20}$$

Therefore (3.12) can be viewed as a toy model for a noncommutative generalisation of the $U(2)\times U(2)$ ABJM model. It would be interesting to construct an $\mathcal{N}=6$ supersymmetric generalisation of (3.12) and to identify a brane construction with magnetic fields in M theory whose low-energy dynamics is related to the generalised model in the appropriate limit [1]. We note that some relevant supergravity backgrounds with magnetic fields have been constructed in the context of the AdS/CFT correspondences for noncommutative $\mathcal{N}=4$ supersymmetric Yang-Mills theory [33] and for various noncommutative generalisations of ABJM models [38] at large N.

4 Noncommutative Chern-Simons Vortex Solitons

Consider the Chern-Simons-Higgs model of the previous section, with Higgs fields in the bifundamental representation of $U(1,1)_\star \times U(1,1)_\star$ and the relativistic sixth order Higgs potential (3.10) [4]. The action is

$$\begin{aligned} S &= S_{CS}^L + S_{CS}^R + S_{Higgs} \\ &= \int dt 2\pi\theta \mathrm{Tr}\frac{\kappa}{2}\left[\epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu^L\partial_\nu A_\rho^L - \frac{2i}{3}A_\mu^L A_\nu^L A_\rho^L\right) - \epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu^R\partial_\nu A_\rho^R - \frac{2i}{3}A_\mu^R A_\nu^R A_\rho^R\right)\right] \\ &\quad + \int dt 2\pi\theta \mathrm{Trtr}\left[D_\mu\phi(D^\mu\phi)^\dagger + V(|\phi|^2)\right], \end{aligned} \tag{4.1}$$

where we introduced the notation $|O|^2 = OO^\dagger$ for any operator O and the covariant derivatives are defined in (3.4), (3.6) and (3.8) depending on the chosen representation for ϕ. The Bogomol'nyi bound and BPS equations for this system can be found by standard methods [32]. The total energy is given by

$$E = \int dt 2\pi\theta \mathrm{Trtr}\left[D_0\phi(D_0\phi)^\dagger + D_i\phi(D_i\phi)^\dagger + V(|\phi|^2)\right], \tag{4.2}$$

which can be written equivalently as

$$\begin{aligned} E &= \int dt 2\pi\theta \mathrm{Trtr}\Bigg[\left|D_0\phi \pm \frac{i}{\kappa}\left(v^2 - |\phi|^2\right)\phi\right|^2 + |(D_1 \pm iD_2)\phi|^2 \\ &\qquad \pm \left(|\phi|^2 - v^2\right)\left(B - \frac{1}{\kappa}j_0\right) \mp \frac{1}{2}\epsilon_{kl}D_k j_l \pm v^2 B\Bigg]. \end{aligned} \tag{4.3}$$

Here j_μ is the current density associated with $U(1)_\star$ rotations,

$$j_\mu = i\left[D_\mu\phi\phi^\dagger - \phi(D_\mu\phi)^\dagger\right] \tag{4.4}$$

and $B = F_{12}$ is the magnetic field. The Gauss law constraint reads

$$\kappa B = j_0 \tag{4.5}$$

so that, up to total derivative terms,

$$E = \int dt 2\pi\theta \mathrm{Trtr}\left[\left|D_0\phi \pm \frac{i}{\kappa}\left(v^2 - |\phi|^2\right)\phi\right|^2 + |(D_1 \pm iD_2)\phi|^2 \pm v^2 B\right]. \tag{4.6}$$

[4] Noncommutative generalisations of the nonrelativistic Jackiw-Pi potential [23] can also be considered, as was done in [21] for the noncommutative generalisation of the abelian Yang-Mills-Higgs system and in [13] for the noncommutative Chern-Simons-Higgs system.

The Bogomol'nyi bound is thus

$$E \geq v^2 \left|\Phi\right|, \tag{4.7}$$

where Φ denotes the total flux density:

$$\Phi = \theta \mathrm{Trtr} B. \tag{4.8}$$

The bound (4.7) is saturated by field configurations satisfying the BPS equations

$$\begin{aligned} (D_1 \pm i D_2)\,\phi &= 0 & (4.9)\\ D_0\phi \pm \frac{i}{\kappa}\left(v^2 - |\phi|^2\right)\phi &= 0. & (4.10) \end{aligned}$$

Using the Gauss law (4.5), eq. (4.10) can be rewritten as

$$B = \pm \frac{2}{\kappa^2}|\phi|^2\left(v^2 - |\phi|^2\right). \tag{4.11}$$

Using the methods of ref. [26, 27, 13], it is now straightforward to find nontopological soliton solutions of eq. (4.9) and (4.10). In the Fock basis, define projection and shift operators P_m and S_m by

$$\begin{aligned} S_m &= \sum_{n=0}^{\infty} |n+m\rangle\langle n| & (4.12)\\ P_m &= \sum_{n=0}^{m-1} |n\rangle\langle n|, & (4.13) \end{aligned}$$

together with a further operator Z_m:

$$Z_m = P_m a P_m + S_m a S_m^\dagger, \tag{4.14}$$

where a denotes the annihilation operator defined in eq. (1.5). The shift operator is a non-unitary isometry,

$$S_m^\dagger S_m = 1, \qquad S_m S_m^\dagger = 1 - P_m \tag{4.15}$$

and it follows that the operator Z_m satisfies the commutation relations

$$\left[Z_m, Z_m^\dagger\right] = 1 - m|m-1\rangle\langle m-1|. \tag{4.16}$$

It will also be useful to introduce the covariant position operators defined by [39, 40]

$$X_i^L = x_i - \theta\epsilon_{ij}A_j^L, \qquad X_i^R = x_i - \theta\epsilon_{ij}A_j^R, \tag{4.17}$$

as they transform covariantly under the gauge transformations of the Chern-Simons fields A^L and A^R. In particular, they can be used to measure the invariant size Δ of the solitons. For any density ρ (e. g. of energy or charge) associated with the solution, a convenient definition is

$$\Delta \equiv \sqrt{\frac{\mathrm{Trtr}\,(X_i - R_i)\,(X_i - R_i)}{\mathrm{Trtr}\rho}}, \tag{4.18}$$

where the R_i denotes the density center defined by

$$R_i \equiv \frac{\mathrm{Trtr} X_i \rho}{\mathrm{Trtr}\rho}. \tag{4.19}$$

Now define operators

$$\begin{aligned} K^L &\equiv \frac{1}{\sqrt{2\theta}}\left(X_1^L - iX_2^L\right) \\ K^R &\equiv \frac{1}{\sqrt{2\theta}}\left(X_1^R - iX_2^R\right). \end{aligned} \tag{4.20}$$

Then

$$B = \frac{1}{\theta}\left(i + \frac{1}{\theta}\left[X_1^L, X_2^L\right]\right) \tag{4.21}$$

and the BPS equations (4.9) and (4.10) can be written equivalently as

$$(K^L)^\dagger \phi - \phi a^\dagger = 0 \tag{4.22}$$

$$1 - \left[K^L, (K^L)^\dagger\right] = \pm\frac{2\theta}{\kappa^2}|\phi|^2(v^2 - |\phi|^2) \tag{4.23}$$

for the left module, and similarly for the right module. For each value of the magnetic field there are two types of solutions to these equations which can be written as

$$\begin{aligned} \phi^L &= \lambda_\pm |m-1\rangle\langle 0|, \qquad & K^L = Z_m \\ \phi^R &= \chi_\pm |m-1\rangle\langle 0|, \qquad & K^R = Z_m, \end{aligned} \tag{4.24}$$

where the $\lambda_\pm$ and $\chi_\pm$ are given by

$$\lambda_\pm^2 = v^2\left[1 \pm \left(1 - \frac{2m\kappa}{\theta v^4}\right)^{\frac{1}{2}}\right]. \tag{4.25}$$

The solutions exist provided

$$\theta \geq \frac{2m\kappa}{v^4} \tag{4.26}$$

and in particular they admit no commutative limits as the noncommutativity parameter $\theta \to 0$. Once the BPS equations are solved, explicit expressions for the gauge fields components A^L_μ and A^R_μ can be given. First, A^R_0 and A^R_0 are determined by (4.10) and one finds

$$A^L_0 = \mp\frac{1}{|\kappa|}\left(v^2 - |\phi^L|^2\right), \qquad A^R_0 = \mp\frac{1}{|\kappa|}\left(v^2 - |\phi^R|^2\right); \tag{4.27}$$

note that these components are also determined by their field equations as derived from action (4.1), viz.

$$\kappa\epsilon_{lk}F^L_{0l} = j^L_k \tag{4.28}$$

and

$$\kappa\epsilon_{lk}F^R_{0k} = j^R_k \tag{4.29}$$

, together with expression (4.4) for the current. The spatial components of A^L_i and A^R_i for the solutions (4.24) are then readily obtained from definitions (4.17) and (4.20):

$$\begin{aligned} A^L_1 &= \frac{i}{\sqrt{2\theta}}\left[P_m\left(a+a^\dagger\right)P_m + S_m\left(a+a^\dagger\right)S^\dagger_m - \left(a+a^\dagger\right)\right] \\ A^L_2 &= \frac{1}{\sqrt{2\theta}}\left[P_m\left(a-a^\dagger\right)P_m + S_m\left(a-a^\dagger\right)S^\dagger_m - \left(a-a^\dagger\right)\right] \end{aligned} \tag{4.30}$$

and similarly for the A^R_i. The operator Z_m is given explicitly by

$$Z_m = \sum_{n=0}^{m-2}\sqrt{n+1}|n\rangle\langle n+1| + \sum_{n=0}^{\infty}\sqrt{n+1}|n+m\rangle\langle n+m+1|. \tag{4.31}$$

The total flux of the solutions is obtained from (4.8) and (4.11); utilising (4.25), one finds

$$\Phi = m. \tag{4.32}$$

The corresponding value of the energy (which saturates the bound (4.7)) are

$$E = mv^2. \tag{4.33}$$

The centre position associated with the Hamiltonian density $\mathcal{H}$ (cf. eq. (4.2) vanishes on the solution,

$$R^H_i = \frac{\mathrm{Trtr} X_i\mathcal{H}_0}{\mathrm{Trtr}\rho} = 0, \tag{4.34}$$

and the size of the energy density distribution is

$$\Delta_{\mathcal{H}} = \sqrt{\theta(m-1)\left(1 + \frac{(m-2)\lambda^2_\pm}{mv^2}\right)}. \tag{4.35}$$

The total charge carried by the solutions is

$$Q = 2\pi\theta \mathrm{Trtr} j_0 = 2\pi\kappa m, \tag{4.36}$$

the centre position R_i^Q associated with the charge density j_0 nanishes on the solution,

$$R_i = \frac{\mathrm{Trtr} X_i j_0}{\mathrm{Trtr} j_0} = 0, \tag{4.37}$$

and the associated size is

$$\Delta_Q = \sqrt{\theta(m-1)}. \tag{4.38}$$

The total angular momentum can be found by applying the Noether method to the action (4.1), which is symmetric under the covariant rotations [41]

$$\begin{aligned} \delta X &= \frac{i}{2\theta}[X_k X_k, X_i] - \epsilon_{ij} X_i \\ \delta A_0 &= -\frac{1}{2\theta} D_0 \left(X_k X_k\right) \\ \delta\phi &= -\frac{i}{2\theta}\left(X_k X_k \phi - \phi x_k x_k\right) \end{aligned} \tag{4.39}$$

The result is [40, 41]

$$Q_J = 2\pi\theta \mathrm{Trtr}\left(\epsilon_{ij} X_i^L T_L^{0j} + \epsilon_{ij} X_i^R T_R^{0j} - \frac{i\theta}{2}\left[D_i D_0 \phi (D_i\phi)^\dagger - D_i\phi(D_i D_0\phi)^\dagger\right]\right) \tag{4.40}$$

where T^{0i} is the stress-energy density

$$T^{0i} = -\frac{1}{2}\left(D_i\phi(D_0\phi)^\dagger + D_0\phi(D_i\phi)^\dagger\right). \tag{4.41}$$

For the solutions (4.24), one finds

$$Q_J = \pi\kappa m(m-2) \tag{4.42}$$

and

$$\Delta_J = \dots \tag{4.43}$$

5 Gravitational Intepretation of the Solutions

Given a field configuration which solves the field equations of our noncommutative Chern-Simons-Higgs model, one can regard it as a solution of the theory of noncommutative gravity with torsion coupled to Chern-Simons gauge fields and charged Higgs fields defined by the action (4.1). Explicitly, the dreibein and connection are obtained by inverting (2.5), which yields the formulae

$$\omega = \frac{1}{2}\left(A^L + A^R\right) \tag{5.1}$$

$$e = \frac{l}{2}\left(A^L - A^R\right). \tag{5.2}$$

Defining the noncommutative metric $G_{\mu\nu}$ by [8, 9]

$$G_{\mu\nu} \equiv e^a_\mu \star e^b_\nu \eta_{ab} \equiv g_{\mu\nu} + i b_{\mu\nu}, \tag{5.3}$$

the second order formalism can be given in terms of the symmetric metric $g_{\mu\nu}$ and anti-symmetric 2-form potential $b_{\mu\nu}$.

References

[1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[2] O. Andreev and H. Dorn, *On open string sigma-model and noncommutative gauge fields*, Phys. Lett. B **476** (2000) 402 [arXiv:hep-th/9912070].

[3] O. Andreev and H. Dorn, *Diagrams of noncommutative Phi**3 theory from string theory*, Nucl. Phys. B **583** (2000) 145 [arXiv:hep-th/0003113].

[4] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[5] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, JHEP **0005** (2000) 020 [arXiv:hep-th/0003160].

[6] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, JHEP **0104** (2001) 001 [arXiv:hep-th/0009142].

[7] M. Banados, O. Chandia, N. E. Grandi, F. A. Schaposnik and G. A. Silva, *Three-dimensional noncommutative gravity*, Phys. Rev. D **64**, 084012 (2001) [arXiv:hep-th/0104264].

[8] J. Wess, *Deformed gravity*, Prog. Theor. Phys. Suppl. **171**, 1 (2007).

[9] J. Wess, *Einstein gravity on deformed spaces*, Lect. Notes Phys. **774**, 39 (2009).

[10] V. P. Nair, *Gravitational fields on a noncommutative space*, Nucl. Phys. B **651**, 313 (2003) [arXiv:hep-th/0112114].

[11] S. Cacciatori, A. H. Chamseddine, D. Klemm, L. Martucci, W. A. Sabra and D. Zanon, *Noncommutative gravity in two dimensions*, Class. Quant. Grav. **19**, 4029 (2002) [arXiv:hep-th/0203038].

[12] S. Cacciatori, D. Klemm, L. Martucci and D. Zanon, *Noncommutative Einstein-AdS gravity in three dimensions*, Phys. Lett. B **536**, 101 (2002) [arXiv:hep-th/0201103].

[13] D. Bak, S. K. Kim, K. S. Soh and J. H. Yee, *Noncommutative Chern-Simons solitons*, Phys. Rev. D **64**, 025018 (2001) [arXiv:hep-th/0102137].

[14] A. P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, Phys. Lett. B **495**, 407 (2000) [arXiv:hep-th/0007043].

[15] A. Khare and M. B. Paranjape, *Solitons in 2+1 dimensional non-commutative Maxwell Chern-Simons Higgs theories*, JHEP **0104**, 002 (2001) [arXiv:hep-th/0102016].

[16] S. Cacciatori and L. Martucci, *Noncommutative AdS supergravity in three dimensions*, Phys. Lett. B **542**, 268 (2002) [arXiv:hep-th/0204152].

[17] L. Alvarez-Gaume and S. R. Wadia, *Gauge theory on a quantum phase space*, Phys. Lett. B **501**, 319 (2001) [arXiv:hep-th/0006219].

[18] D. Tong, *TASI lectures on solitons*, [arXiv:hep-th/0509216].

[19] A. Hanany and D. Tong, *Vortices, instantons and branes*, JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150].

[20] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, Nucl. Phys. B **311**, 46 (1988).

[21] D. P. Jatkar, G. Mandal and S. R. Wadia, *Nielsen-Olesen vortices in noncommutative Abelian Higgs model*, JHEP **0009** (2000) 018 [arXiv:hep-th/0007078].

[22] D. Bak, *Exact multi-vortex solutions in noncommutative Abelian-Higgs theory*, Phys. Lett. B **495** (2000) 251 [arXiv:hep-th/0008204].

[23] R. Jackiw and S. Y. Pi, *Classical and quantal nonrelativistic Chern-Simons theory*, Phys. Rev. D **42** (1990) 3500 [Erratum-ibid. D **48** (1993) 3929].

[24] V. P. Nair and A. P. Polychronakos, *On level quantization for the noncommutative Chern-Simons theory*, Phys. Rev. Lett. **87** (2001) 030403 [arXiv:hep-th/0102181].

[25] D. Bak, K. M. Lee and J. H. Park, *Chern-Simons theories on noncommutative plane*, Phys. Rev. Lett. **87** (2001) 030402 [arXiv:hep-th/0102188].

[26] J. A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, JHEP **0012** (2000) 024 [arXiv:hep-th/0010060].

[27] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, *D-branes and strings as non-commutative solitons*, JHEP **0007** (2000) 042 [arXiv:hep-th/0005031].

[28] A. Achucarro and P. K. Townsend, *A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories*, Phys. Lett. B **180** (1986) 89.

[29] R. Jackiw, K. M. Lee and E. J. Weinberg, Selfdual Chern-Simons solitons, Phys. Rev. D **42** (1990) 3488.

[30] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **0810** (2008) 091 [arXiv:hep-th/0806.1218].

[31] J. A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, [arXiv:hep-th/0102076].

[32] E. B. Bogomoln'yi, *The Stability of Classical Solutions*, Sov. J. Nucl. Phys., **24**(1976) 49.

[33] J. M. Maldacena and J. G. Russo, *Large N limit of non-commutative gauge theories*, JHEP **9909** (1999) 025 [arXiv:hep-th/9908134].

[34] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, *A Massive Study of M2-brane Proposals*, JHEP **0809** (2008) 113 [arXiv:hep-th/0807.1074].

[35] C. Kim, Y. Kim, O. K. Kwon and H. Nakajima, *Vortex-type Half-BPS Solitons in ABJM Theory*, Phys. Rev. D **80** (2009) 045013 [arXiv:hep-th/0905.1759].

[36] T. Asakawa and S. Kobayashi, *Noncommutative Solitons of Gravity*, Class. Quant. Grav. **27**, 105014 (2010) [arXiv:hep-th/0911.2136].

[37] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park and P. Yi, *A Nonperturbative Test of M2-Brane Theory*, JHEP **0811** (2008) 058 [arXiv:hep-th/0809.1771].

[38] E. Imeroni, *On deformed gauge theories and their string/M-theory duals*, JHEP **0810** (2008) 026 [arXiv:hep-th/0808.1271].

[39] D. J. Gross, A. Hashimoto and N. Itzhaki, *Observables of non-commutative gauge theories*, Adv. Theor. Math. Phys. **4** (2000) 893 [arXiv:hep-th/0008075].

[40] D. Bak, K. M. Lee and J. H. Park, *Comments on noncommutative gauge theories*, Phys. Lett. B **501** (2001) 305 [arXiv:hep-th/0011244].

[41] R. Jackiw, *Gauge Covariant Conformal Transformations*, Phys. Rev. Lett. **41** (1978) 1635.